# The most interesting surface maps

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## History

In the 1920s and 1930s, J. Nielsen initiated a program to classify homeomorphisms of surfaces (up to isotopy). A complete classification was obtained by Thurston in the 1970s. Thus began an active research area in the study of surface homeomorphisms, especially pseudo-Anosov homeomorphisms.

## Surface homeomorphisms

Let $S_{g,n}$ be a hyperbolic surface with genus $g$ and $n$ marked points.

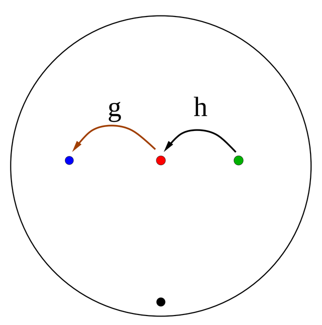
### Nielsen-Thurston Classification

Any homeomorphism $f : S \to S$ is isotopic to either a finite order, reducible, or pseudo-Anosov homeomorphism.

1. **Finite order homeomorphism**
   If there exists $k$ such that $f^k$ =identity, then $f$ is finite order.

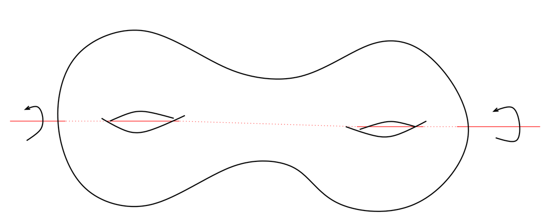
Examples:

**1.**




f = h g

**2.**

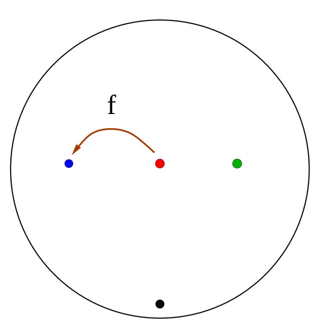


f= a rotation around the skew

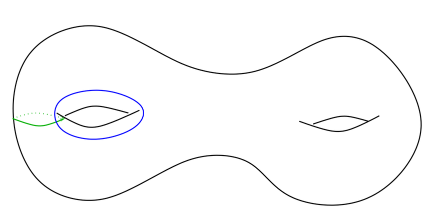
2. **Reducible homeomorphism**
   If there exists a reducing system $\alpha = \sqcup_{i=1}^{k} \alpha_i$ with $f(\alpha) = \alpha$, then $f$ is reducible.

Examples:

**1.**




**2.**

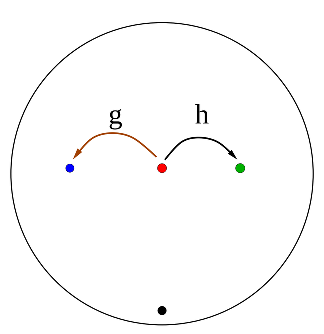


f= right Dehn twist on green then left Dehn twist on blue

3. **Pseudo-Anosov homeomorphism**
   If f is infinite order and irreducible, then $f$ is pseudo-Anosov.

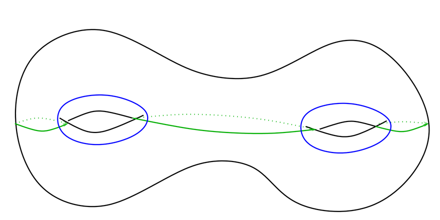
Examples:

**1.**




f = h g

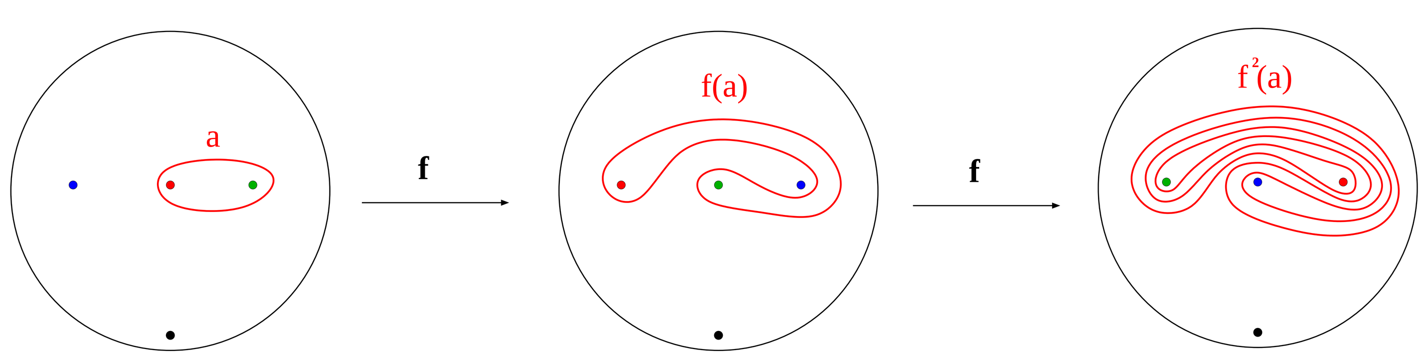
**2.**



f= right Dehn twist on green then left Dehn twist on blue

### What do pseudo-Anosov homeomorphisms do to surfaces?

Choose a simple closed curve $a$ and look at what happens after we apply $f$ to $a$.

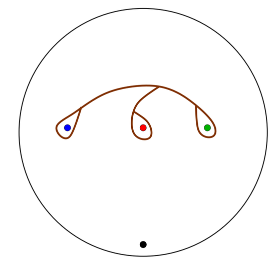



Notice that it is not easy to draw $f^5(a)$ directly.

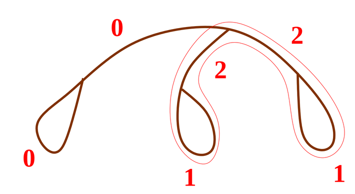
## Tools for understanding pseudo-Anosov homeomorphisms

### Thurston's train tracks

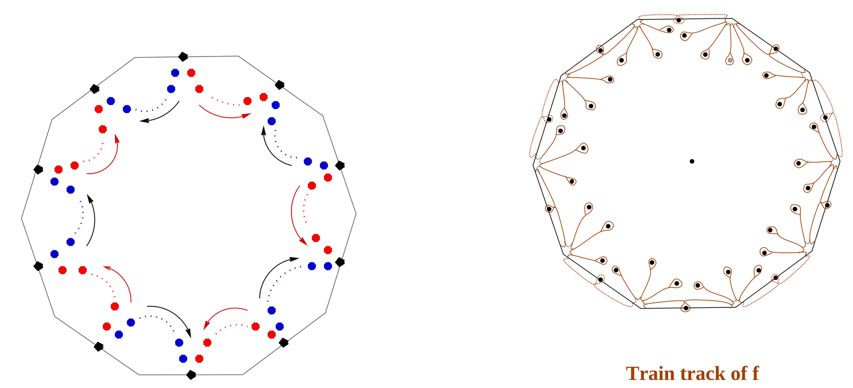
A train track $\tau$ of $f : S \to S$ is a "smooth one-complex" embedded in $S$ with some non-degeneracy conditions. Below is an example of a train track.



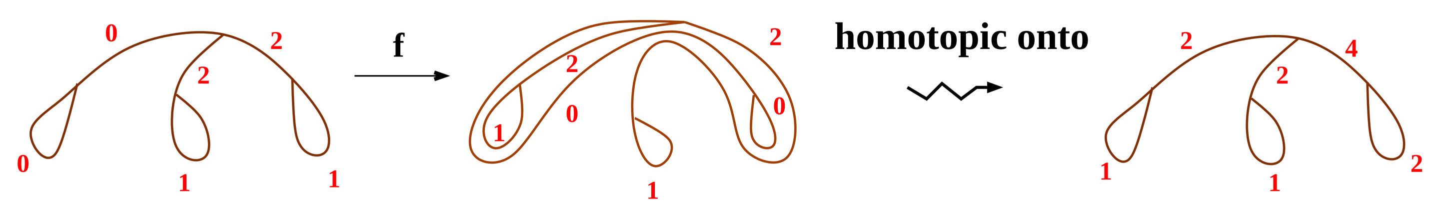
If $f : S \to S$ is pseudo-Anosov, then there exists a train track $\tau$ such that "$f(\tau)$ collapses onto $\tau$". Note that $f$ in Example 1 of pseudo-Anosov homeomorphisms (in the left column) has the train track above. Train tracks can describe curves by weights. Here is an example. The simple closed curve $a$ determines weights on the train track in the following way: If $a$ has two arcs passing through a branch, then it has weight 2.



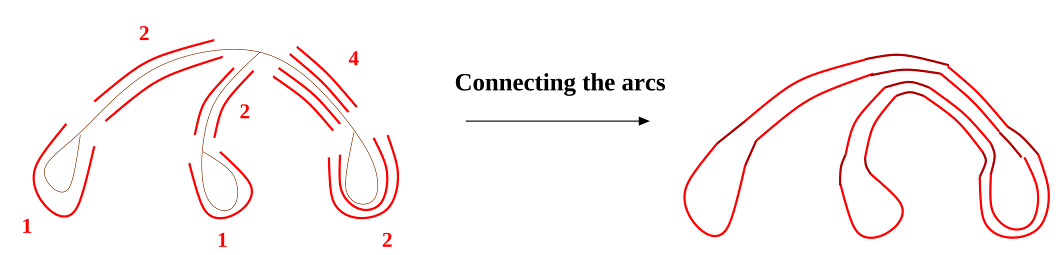

**Remark:** We can use train tracks to find the action on curves.
As we saw earlier, even when $f$ is "simple", it is hard to directly keep track of what happens to the curve $a$ under iteration of $f$. Most of the time pseudo-Anosov homeomorphisms are much more complicated. For example:
$f : S_{2,n} \to S_{2,n}$ (following the red arrows then the black arrows)




Let's see how we can use the train track. First, the simple closed curve $a$ determines weights on the train track. After we apply $f$, we obtain the new weights on the train track.




From the new weights we can recover $f(a)$. The weights on the branches tell us the number of arcs, which we then connect by following the tracks.




### Dilatations of pseudo-Anosov homeomorphisms

**Theorem:(Thurston)**
For any hyperbolic metric on $S_{g,n}$, and any pseudo-Anosov $f : S_{g,n} \to S_{g,n}$, there exists $\lambda(f)$ such that

$$\lim_{k\to\infty} \sqrt[k]{length(f^k(a))} = \lambda(f),$$

for any nontrival closed curve $a$.

We call $\lambda(f)$ the **dilatation** of the pseudo-Anosov map $f$.

**Facts:**
1. $\lambda(f) > 1$.
2. $\lambda(f)$ is the growth rate of the length of closed curves.
3. If $f$ is isotopic to $g$, then $\lambda(f) = \lambda(g)$.
4. $\{\lambda(f) | f : S_{g,n} \to S_{g,n}$ pseudo-Anosov$\}$ is discrete (Arnoux-Yoccoz and Ivanov).

## The asymptotic behavior of least pA dilatations

Since the set of pseudo-Anosov dilatations is discrete for each $g$ and $n$, there is a minimal element of the set. We call it the least pseudo-Anosov dilatation, which is defined by

$$l_{g,n} := \min\left\{ \log \lambda(f) \middle| f : S_{g,n} \to S_{g,n} \text{ pseudo-Anosov} \right\}.$$

### Known results

- **For closed surfaces**
  In 1991, Penner proves

$$\frac{\log 2}{12g-12} \le l_{g,0} \le \frac{\log 11}{g}, \text{ for } g \ge 2.$$

  These bounds has been improved by Bauer (1992), McMullen (2000), Minakawa (2006), and Hironaka-Kin (2006). The best known bounds are

$$\frac{\log 2}{6g-6} \le l_{g,0} \le \frac{\log(2+\sqrt{3})}{g}, \text{ for } g \ge 2.$$

- **For surfaces with marked points**
  In the same paper, Penner proves a lower bound for surfaces with marked points,

$$l_{g,n} \ge \frac{\log 2}{12g+4n-12}, \text{ for } 3g+n-3>0.$$

  He seems to suggest that there should be an analogous upper bound.
- **For spheres with marked points**
  Hironaka-Kin in 2006 show that Penner's conjecture holds for the sphere case by finding an upper bound

$$l_{0,n} < \frac{2\log(2+\sqrt{3})}{n-3}, \text{ for } n \ge 4.$$

  We can combine it with Penner's lower bound,

$$\frac{\log 2}{4n-12} \le l_{0,n} < \frac{2\log(2+\sqrt{3})}{n-3}, \text{ for } n \ge 4.$$

**Remark:** The topology of a surface $S$ is uniquely determined by its Euler characteristic $\chi(S_{g,n}) = 2-2g-n$. These results show that if we change the topology of the underlying surface, the least pseudo-Anosov dilatations will change. In particular, in the known cases it is proportional to $\frac{1}{\chi(S)}$.

**Question:** Does $l_{g,n}$ always go to 0 on the order of $\frac{1}{\chi(S)}$, which is the same as $\frac{1}{g+n}$? No!

## The main theorem (Tsai, 2008)

Given genus $g \ge 2$, there is a constant $c_g$, depending on $g$, such that

$$\frac{\log n}{c_g n} < l_{g,n} < \frac{c_g \log n}{n},$$

for all $n \ge 3$.

**Remark:** For fixed $g \ge 2$, the asymptotic behavior of $l_{g,n}$ is like $\frac{\log n}{n}$, and not $\frac{1}{n}$.

**Remark:** The earlier example of $f : S_{2,n} \to S_{2,n}$ is the example which gives the upper bound of the theorem.

## Ongoing projects

1. Get a smaller constant since our $c_g$ is exponential in $g$.
2. For given $n$, what is the asymptotic behavior of $l_{g,n}$?
3. Find a sequence of pairs of $g$ and $n$, such that $l_{g,n}$ is "large". There are examples of small dilatations for $n = g$, $n = g+1$ and $n = g+2$ where $l_{g,n}$ is on the order of $\frac{1}{g}$. However, the constructions do not work for $n = g-2$.
4. Find a general construction of pseudo-Anosov homeomorphisms with small dilatations.