SEMISIMPLE LIE GROUPS AND RIEMANNIAN SYMMETRIC SPACES.

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1. Introduction.

This article is a set of complimentary notes for the talk given by the author at the Higgs Bundles and Harmonic Maps Workshop in January 2015. Our goal is to provide the reader with a brief summary of some important properties of semisimple Lie groups, and how they relate to Riemannian symmetric spaces. Emphasis will be given to the case when the symmetric space is of non-compact type. To be concise, most of the proofs given here are lacking in detail, and are there mainly to indicate the main ideas used to obtain the stated results. Most of the material here is from Chapter 2 of [1], the first six chapters of [2], and Chapters 2 and 3 of [3].

2. Lie groups introduction.

In this section, we will recall some Lie group basics, starting with the definition.

Definition 2.1.

(1) A *Lie group* G is a real analytic manifold with a group structure such that multiplication and inversion are real analytic diffeomorphisms.
(2) A Lie group G is *complex* if it is also a complex manifold, and multiplication and inversion are holomorphic.
(3) The *Lie algebra* of G, denoted $\mathfrak{g}$, is the tangent space at the identity of G.
(4) Let G_1 and G_2 be two Lie groups. A real analytic map $f : G_1 \to G_2$ is a *Lie group morphism* if f is a group homomorphism and a real analytic immersion.
(5) A *Lie subgroup* H of a Lie group G is a subgroup equipped with a real analytic structure so that the inclusion map of H into G is a morphism of Lie groups.

As of now, $\mathfrak{g}$ is simply a vector space. Shortly, we will equip $\mathfrak{g}$ with an algebraic structure that captures the infinitesimal group structure of G. In the case when G is a complex Lie group, $\mathfrak{g}$ comes equipped with a natural complex structure. A Lie group morphism $f : G_1 \to G_2$ induces a $\mathbb{R}$-linear map $df : \mathfrak{g}_1 \to \mathfrak{g}_2$, which is also $\mathbb{C}$-linear if G_1 and G_2 are complex Lie groups and f is holomorphic.

The simplest example of a Lie group is $\mathbb{R}$ equipped with addition and the usual real analytic structure. To study more general Lie groups, it is often useful to consider Lie group morphisms of $\mathbb{R}$ into G. These are called *one parameter subgroups* of G, and are analogous in Lie theory to geodesics in Riemannian geometry. One can first see this analogy from the following proposition.

Proposition 2.2. *The set of one parameter subgroups in G are in bijection with $\mathfrak{g}$.*

Proof. If γ is a one parameter subgroup, then $\gamma(0) = \mathrm{id}$, so $\gamma'(0) \in \mathfrak{g}$. This gives us a map ϕ from the set of one parameter subgroups to $\mathfrak{g}$. To prove that ϕ is surjective, one needs to construct, for every $X \in \mathfrak{g}$, a one parameter subgroup γ such that $\gamma'(0) = X$. Observe that every vector $X \in \mathfrak{g}$ generates a non-vanishing, left invariant vector field V on G by pushing forward X to every point in G using the left action of G on itself. Let γ be a flow line of V through id, and note that $\gamma'(0) = V(\mathrm{id}) = X$. Moreover, for every $s \in \mathbb{R}$, one can easily use the left invariance of V to show that the curves $\eta(t) := \gamma(s)\gamma(t)$ and $\omega(t) := \gamma(s+t)$ are flow lines of V with the same initial condition. This implies that γ is a one-parameter subgroup, so ϕ is surjective.

To prove ϕ is injective, observe that for any one parameter subgroup γ, the vector field $\gamma'(t)$ along γ has to be invariant under left multiplication by $\gamma(s)$ for $s \in \mathbb{R}$, so it can be extended to a left invariant vector field on G. Thus, it has to be the flow line of the unique left invariant vector field V on G with $V(0) = X$. □

Notation 2.3. By the above proposition, we can use the following notation. For any $X \in \mathfrak{g}$, let γ_X be the one parameter subgroup so that $\gamma_X'(0) = X$.

Definition 2.4. The *exponential map*, $\exp : \mathfrak{g} \to G$, is defined by $\exp(X) = \gamma_X(1)$.

It is clear from this definition that $\gamma_X(t) = \exp(tX)$.

Example 2.5. Let $F = \mathbb{C}, \mathbb{R}$. Here, we list several examples of common Lie groups, along with their Lie algebras, that will be used in the rest of this article.

(1) $G = GL_n(F) := \{M \in M_{n\times n}(F) : M \text{ is invertible}\}$.
$\mathfrak{g} = \mathfrak{gl}_n(F) = M_{n\times n}(F)$.
(2) $G = SL_n(F) := \{M \in GL_n(F) : \det(M) = 1\}$.
$\mathfrak{g} = \mathfrak{sl}_n(F) = \{M \in \mathfrak{sl}_n(F) : \mathrm{tr}\,(M) = 0\}$.
(3) $G = SU(n) := \{M \in SL_n(\mathbb{C}) : MM^* = \mathrm{id}\}$.
$\mathfrak{g} = \mathfrak{su}(n) = \{M \in \mathfrak{sl}_n(\mathbb{C}) : M + M^* = 0\}$.
(4) $G = SO(n) := \{M \in SL_n(\mathbb{R}) : MM^T = \mathrm{id}\}$.
$\mathfrak{g} = \mathfrak{so}(n) = \{M \in \mathfrak{sl}_n(\mathbb{R}) : M + M^T = 0\}$.
(5) $G = B_n(F) := \{M \in SL_n(F) : M \text{ is upper triangular with 1's along the diagonal}\}$.
$\mathfrak{g} = \mathfrak{b}_n(F) = \{M \in \mathfrak{sl}_n(F) : M \text{ is upper triangular with 0's along the diagonal}\}$.

In all of the above, the exponential map is exponentiation of matrices, i.e,

$$\exp(M) = \sum_{i=1}^{\infty} \frac{1}{i!} M^i$$

for all $M \in \mathfrak{g}$.

Next, we will define a natural algebraic structure on the vector space $\mathfrak{g}$, called the Lie bracket. It captures the infinitesimal behavior of the group structure on G.

Definition 2.6.

(1) For all $h \in G$, let $C_h : G \to G$ be conjugation by h. Define the *adjoint* of h, $Ad(h) := (dC_h)_{\mathrm{id}} : \mathfrak{g} \to \mathfrak{g}$.
(2) The *adjoint representation* of G, $Ad : G \to GL(\mathfrak{g})$ is the morphism of Lie groups defined by $Ad : h \mapsto Ad(h)$, and the *adjoint representation* of $\mathfrak{g}$ is the linear map $ad := (dAd)_{\mathrm{id}} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$.
(3) The *Lie bracket* is the bilinear map $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ defined by $[X, Y] = ad(X)Y$.

The next proposition summarizes some basic properties of the bracket on $\mathfrak{g}$.

Proposition 2.7.

(1) *For all $g \in G$ and for all $X \in \mathfrak{g}$, $C_g(\gamma_X(t)) = \gamma_{Ad(g)X}(t)$.*
(2) *For all $X, Y \in \mathfrak{g}$, $[X,Y] = ad(X)Y = \frac{d}{dt}|_{t=0} Ad(\gamma_X(t))Y$.*
(3) *The kernel of ad is $\{X \in \mathfrak{g} : [X,Y] = 0$ for all $Y \in \mathfrak{g}\}$.*
(4) *$[\cdot,\cdot]$ is bilinear ($\mathbb{R}$-bilinear when G is a real Lie group and $\mathbb{C}$-bilinear when G is a complex Lie group.)*
(5) *Let $X, Y \in \mathfrak{g}$. If V_X and V_Y are left invariant vector fields on G so that $V_X(\mathrm{id}) = X$ and $V_Y(\mathrm{id}) = Y$, then $[V_X, V_Y] := V_X V_Y - V_Y V_X$ is a left invariant vector field with $[X,Y] = [V_X, V_Y](\mathrm{id})$.*
(6) *$[X,X] = 0$ for all $X \in \mathfrak{g}$.*
(7) *$[\cdot,\cdot]$ satisfies the* Jacobi identity*, i.e. $ad([X,Y]) = [ad(X), ad(Y)]$ for all $X, Y \in \mathfrak{g}$.*

Proof. Parts (1) to (4) and (6) follow immediately from the definition. The proof of (5) is a short computation involving (1), (2) and the Baker-Campbell-Hausdorff formula for the exponential map. Part (7) is an easy consequence of (5) and the Jacobi identity for vector fields. □

Proposition 2.7 motivates the following definition.

Definition 2.8.

(1) A *Lie algebra* is a vector space (over $\mathbb{C}$ or $\mathbb{R}$) equipped with a bilinear map $[\cdot,\cdot]$, called the *Lie bracket*, so that properties (6) and (7) of Propositon 2.7 hold.
(2) Let $\mathfrak{g}_1$, $\mathfrak{g}_2$ be two Lie algebras. A linear map $f : \mathfrak{g}_1 \to \mathfrak{g}_2$ is a *Lie algebra morphism* if $[f(X), f(Y)] = f([X,Y])$ for all $X, Y \in \mathfrak{g}_1$.
(3) A *Lie subalgebra* $\mathfrak{h}$ of a Lie algebra $\mathfrak{g}$ is a vector subspace that is closed under the Lie bracket on $\mathfrak{g}$.

If a Lie algebra arises as the Lie algebra of a Lie group, it captures a surprising amount of information about the Lie group. At the same time, it is a much simpler object to study since it is a linear space. In the rest of this section, we will outline some key results in the relationship between Lie algebras and Lie groups.

Theorem 2.9.

(1) *If $f : G_1 \to G_2$ is a morphism of Lie groups, then $df_{\mathrm{id}} : \mathfrak{g}_1 \to \mathfrak{g}_2$ is a morphism of Lie algebras.*
(2) *The set of connected Lie subgroups of G is in bijection with the set of Lie algebras in $\mathfrak{g}$.*

Proof. Proof of (1). Let X and Y be elements in $\mathfrak{g}_1$ and let V_X and V_Y be the left invariant vector fields on G_1 so that $V_X(\mathrm{id}) = X$ and $V_Y(\mathrm{id}) = Y$. An easy computation shows that $df(V_X)$ and $df(V_Y)$ are the left invariant vector fields so that $df(V_X)(\mathrm{id}) = df_{\mathrm{id}}X$ and $df(V_Y)(\mathrm{id}) = df_{\mathrm{id}}Y$. Part (1) then follows from the fact that the push forward of diffeomorphisms preserve the bracket on vector fields.

Proof of (2). Given a Lie subgroup H of G, the inclusion map of H into G induces an injective Lie algebra morphism of $\mathfrak{h}$ into $\mathfrak{g}$. On the other hand, if $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$, then the left G action on itself induces a left invariant subbundle $V_{\mathfrak{h}}$ of TG so that $V_{\mathfrak{h}}(\mathrm{id}) = \mathfrak{h}$. Part (5) of Proposition 2.7 imply that this distribution is closed under the bracket on vector fields, so the Frobenius theorem tells us that there is an immersed submanifold H of G whose tangent bundle is $V_{\mathfrak{h}}$. For all $h \in H$, $h \cdot H$ is also a submanifold whose tangent bundle is also $V_{\mathfrak{h}}$ because $\mathfrak{h}$ is invariant under $Ad(h)$. Thus, $H = h \cdot H$. Similarly, one can show that H is closed under inversion, so H must be a group. □

Theorem 2.10 (Lie's third theorem). *Any Lie algebra is the Lie algebra of a Lie group.*

Proof. Follows from (2) of Proposition 2.9 and Ado's Theorem, which states that every Lie algebra can be embedded as a Lie subalgebra of $\mathfrak{gl}(n, \mathbb{R})$. □

Example 2.11. Let G be a linear group. For all $g \in G$ and for all $X, Y \in \mathfrak{g}$, $Ad(g)X = gXg^{-1}$ and $[X, Y] = XY - YX$.

3. Semisimple Lie algebras.

Since we are interested in Riemannian symmetric spaces, we will now specialize to the case of semisimple Lie groups. Later, we will see that these arise as the connected components isometry groups of Riemannian symmetric spaces.

Definition 3.1. Let G be a Lie group and $\mathfrak{g}$ its Lie algebra.

(1) An *ideal* $\mathfrak{i} \subset \mathfrak{g}$ is a vector subspace such that for all $X \in \mathfrak{i}$ and for all $Y \in \mathfrak{g}$, we have $[X, Y] \in \mathfrak{i}$.
(2) $\mathfrak{g}$ is *abelian* if $[X, Y] = 0$ for all $X, Y \in \mathfrak{g}$.
(3) $\mathfrak{g}$ is *simple* if $\mathfrak{g}$ is not abelian and has no non-trivial ideals.
(4) $\mathfrak{g}$ is *semisimple* if $\mathfrak{g} = \mathfrak{g}_1 \times \cdots \times \mathfrak{g}_n$ (as Lie algebras), where each $\mathfrak{g}_i$ is simple.
(5) G is *semisimple* if $\mathfrak{g}$ is semisimple.
(6) $\mathfrak{g}$ is *compact* if the Lie group $Int(\mathfrak{g}) \subset GL(\mathfrak{g})$ corresponding to $ad(\mathfrak{g}) \subset \mathfrak{gl}(\mathfrak{g})$ is compact.

Example 3.2.

(1) $\mathfrak{sl}_n(F)$, $\mathfrak{su}(n)$ and $\mathfrak{so}(n)$ are semisimple.
(2) $\mathfrak{gl}_n(F)$ is not semisimple because

$$\mathfrak{gl}_n(F) = Span_F \left\{ \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \right\} \times \mathfrak{sl}_n(F)$$

(3) $\mathfrak{b}_n(F)$ is not semisimple because it cannot be decomposed as a product of Lie algebras, and the subset of matrices in $\mathfrak{b}_n(F)$ with 0 everywhere except for the upper left hand corner is an ideal.
(4) $\mathfrak{su}(n)$ and $\mathfrak{so}(n)$ are compact.

On any Lie algebra, there is a canonical bilinear form we can define, called the Killing form.

Definition 3.3. The *Killing form* on $\mathfrak{g}$ is the bilinear form B given by

$$B(X, Y) = \operatorname{tr}(ad(X) \circ ad(Y)).$$

When $\mathfrak{g}$ is a complex Lie algebra, then the Killing form is a complex valued form. However, by forgetting the complex structure, we can also realize $\mathfrak{g}$ as a real Lie algebra, in which case the Killing form is a real valued form. In situations where there might be some ambiguity, we will refer to the complex valued Killing form as the *complex Killing form*, and the real valued Killing form as the *real Killing form*. One can check that the real Killing form is twice the real part of the complex Killing form.

One useful property of the Killing form is that it detects whether a Lie algebra is semisimple. Furthermore, if the Lie algebra is semisimple, the Killing form can also detect if it is compact. These, and other properties of the Killing form are summarized in the next proposition.

Proposition 3.4.

(1) $B([X,Y],Z) = B(X,[Y,Z])$ *for all* $X,Y,Z \in \mathfrak{g}$.
(2) *If* $\mathfrak{g} = \mathfrak{g}_1 \times \cdots \times \mathfrak{g}_n$*, then* $B_{\mathfrak{g}} = B_{\mathfrak{g}_1} \times \cdots \times B_{\mathfrak{g}_n}$.
(3) *If* $\mathfrak{i} \subset \mathfrak{g}$ *is an ideal, then* $B_{\mathfrak{i}} = B_{\mathfrak{g}}|_{\mathfrak{i}\times\mathfrak{i}}$.
(4) $\mathfrak{g}$ *is semisimple if and only if* B *is non-degenerate.*
(5) *If* $\mathfrak{g}$ *is semisimple, then* $\mathfrak{g}$ *is compact if and only if* B *is negative definite.*

Proof. Part (1) follows from the Jacobi identity and the fact that $\mathrm{tr}\,(XY) = \mathrm{tr}\,(YX)$ for any $X, Y \in \mathfrak{gl}(\mathfrak{g})$, and (2) holds because by definition, if $X \in \mathfrak{g}_i$ and $Y \in \mathfrak{g}_j$ and $i \neq j$, then $[X,Y] = 0$. Part (3) is an easy consequence of the linear algebra fact that if W is a subspace of a vector space V and ϕ is an endomorphism with image W, then $\mathrm{tr}\,(\phi|_W) = \mathrm{tr}\,(\phi)$.

Proof of (4). Observe that (1) implies

$$\{X \in \mathfrak{g} : B(X,Y) = 0 \text{ for all } Y \in \mathfrak{g}\}$$

is an ideal. Hence, if $\mathfrak{g}$ is simple, then B has to be non-degenerate. Part (2) then implies the same has to be true in the case when $\mathfrak{g}$ is semisimple. On the other hand, if B is non-degenerate and $\mathfrak{a} \subset \mathfrak{g}$ is an ideal, then one can show use (1) to show that $\mathfrak{a}^\perp \subset \mathfrak{g}$ is also an ideal, and (3) implies that $B|_{\mathfrak{a}\times\mathfrak{a}}$ and $B|_{\mathfrak{a}^\perp\times\mathfrak{a}^\perp}$ are non-degenerate. Applying this iteratively allows us to decompose $\mathfrak{g}$ into a product of simple Lie algebras, so $\mathfrak{g}$ is semisimple.

Proof of (5). Suppose that B is negative definite. Since $Int(\mathfrak{g})$ preserves B, it is a closed subgroup of some orthogonal subgroup of $GL(\mathfrak{g})$ and thus is compact. On the other hand, if $Int(\mathfrak{g})$ is compact, then it lies in some orthogonal subgroup of $GL(\mathfrak{g})$. This means that we can choose a basis of $\mathfrak{g}$ so that every element in $Int(\mathfrak{g})$ is an orthogonal matrix and hence every element in $ad(\mathfrak{g})$ is skew-symmetric in this basis. For any $X \in \mathfrak{g}$, let $(a_{i,j})_{n\times n}$ is the skew-symmetric matrix representing $ad(X)$ in this basis. Then

$$B(X,X) = \sum_{i,j} a_{i,j}a_{j,i} = -\sum_{i,j} a_{i,j}^2 \leq 0$$

and equality holds if and only if X lies in the center of $\mathfrak{g}$. Since $\mathfrak{g}$ is semisimple, its center is trivial, so B is negative definite.

□

Example 3.5.

(1) On $\mathfrak{gl}_n(F)$, $B(X,Y) = 2n\mathrm{tr}\,(XY) - 2\mathrm{tr}\,(X)\mathrm{tr}\,(Y)$ for all $X, Y \in \mathfrak{gl}_n(F)$. Hence, B is degenerate because

$$B\left(\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}, Y\right) = 0 \text{ for all } Y \in \mathfrak{gl}_n(F).$$

(2) On $\mathfrak{b}_n(F)$, $B(X,Y) = n\mathrm{tr}\,(XY)$ for all $X, Y \in \mathfrak{b}_n(F)$. Hence, $B(X,Y) = 0$ for all $X, Y \in \mathfrak{b}_n(F)$ so B is degenerate.

An important feature of any semisimple complex Lie algebra $\mathfrak{g}$ is the existence of a special vector space decomposition of $\mathfrak{g}$ known as the *root space decomposition*. Using this, one can show that any semisimple Lie algebra has an important vector space decomposition known as the *Cartan decomposition*. The rest of this section will be used to explain these two decompositions.

3.1. **Root space decomposition.** The root space decomposition is very useful when studying complex semisimple Lie algebras because the Killing form and Lie bracket behave very well with respect to this decomposition. A surprising amount of information about the structure of the Lie algebra can thus be read off quite easily from this decomposition. However, for our purposes, we will only need the root space decomposition to prove some statements about the Cartan decomposition. As such, our description in this subsection is intentionally sparse, and proofs for the results stated here will not be discussed, but referenced.

The root space decomposition relies on the existence of a non-trivial Cartan subalgebra, which we will now define.

Definition 3.6. Let $\mathfrak{g}$ be a complex semisimple Lie algebra. A *Cartan subalgebra* is a maximal abelian subalgebra $\mathfrak{a} \subset \mathfrak{g}$ such that $ad(\mathfrak{a}) \subset \mathfrak{gl}(\mathfrak{g})$ is diagonalizable.

Note that in the above definition $ad(\mathfrak{a})$ is simultaneously diagonalizable because it is abelian.

Example 3.7.

(1) For $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C}), \mathfrak{gl}_n(\mathbb{C})$ and $\mathfrak{b}_n(\mathbb{C})$, the set of diagonal matrices in $\mathfrak{g}$ is a Cartan subalgebra.
(2) Let $\mathfrak{b}'_n(\mathbb{C})$ be the subset of $\mathfrak{b}_n(\mathbb{C})$ that has zeroes along the diagonal. This is a complex Lie algebra with trivial Cartan subalgebra.

From the above example, one sees that non-trivial Cartan subalgebras need not exist in any complex Lie algebra. However, the next theorem tells us that they do exist if the Lie algebra is semisimple, and in that case, they are unique up to conjugation.

Theorem 3.8. *Every complex semisimple Lie algebra $\mathfrak{g}$ has a non-trivial Cartan subalgebra. Also, for any two Cartan subalgebras $\mathfrak{a}_1, \mathfrak{a}_2 \subset \mathfrak{g}$, there is some $\psi \in Int(\mathfrak{g})$ such that $\psi(\mathfrak{a}_1) = \mathfrak{a}_2$.*

Proof. See Thoerem 3.1 in Chapter III of [2] for a proof of the existence of a non-trivial Cartan subalgebra. For the uniqueness up to conjugation, a proof can be found in Sections 16.2 to 16.4 of [3]. □

Equipped with the above theorem, we can now construct the root space decomposition for any complex semisimple Lie algebra.

Definition 3.9. Let $\mathfrak{a} \subset \mathfrak{g}$ be a Cartan subalgebra.

(1) For any $\alpha \in \mathfrak{a}^*$, define $\mathfrak{g}_\alpha := \{X \in \mathfrak{g} : ad(H) \cdot X = \alpha(H)X \text{ for all } H \in \mathfrak{a}\}$.
(2) Define $\Lambda := \{\alpha \in \mathfrak{a}^* : \mathfrak{g}_\alpha \neq \{0\}\} \setminus \{0\}$.

If $\mathfrak{g}_\alpha \neq \{0\}$, then α is a *root* and $\mathfrak{g}_\alpha$ is called the *root space* of α.

Theorem 3.10. *Let* $\mathfrak{a}$ *be a Cartan subalgebra of a complex semisimple Lie algebra* $\mathfrak{g}$. *Then* $\mathfrak{a} = \mathfrak{g}_0$.

Proof. See Proposition 8.2 of [3] for a proof. □

Definition 3.11. The vector space decomposition of a complex semisimple Lie algebra

$$\mathfrak{g} = \mathfrak{a} \oplus \bigoplus_{\alpha \in \Lambda} \mathfrak{g}_\alpha$$

is the *root space decomposition.*

Example 3.12. Let $\mathfrak{a} \subset \mathfrak{sl}_3(\mathbb{C})$ be the Cartan subalgebra that is the set of diagonal matrices in $\mathfrak{sl}_3(\mathbb{C})$. Then

$$\Lambda := \{\alpha_{i,j} : i, j = 1, 2, 3 \text{ and } i \neq j\}$$

is the set of roots for $\mathfrak{sl}_3(\mathbb{C})$ with respect to $\mathfrak{a}$, where

$$\alpha_{i,j} \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix} = a_i - a_j.$$

The root space $\mathfrak{g}_{\alpha_{i,j}}$ is then the $\mathbb{C}$-span of the matrix $E_{i,j}$ with 1 in the (i, j) entry and zero every where else.

The next theorem summarizes the important properties of this decomposition.

Theorem 3.13.

(1) Λ *spans* $\mathfrak{a}^*$.
(2) $\dim(\mathfrak{g}_\alpha) = 1$ *for all* $\alpha \in \Lambda$.
(3) *If* $\alpha \in \Lambda$, *then* $-\alpha \in \Lambda$.
(4) *If* $\alpha, \alpha' \in \Lambda$ *and* $\alpha \neq -\alpha'$, *then* $B(\mathfrak{g}_\alpha, \mathfrak{g}_{\alpha'}) = 0$ *and* $[\mathfrak{g}_\alpha, \mathfrak{g}_{\alpha'}] = 0$.
(5) $[\mathfrak{g}_\alpha, \mathfrak{g}_{\alpha'}] \subset \mathfrak{g}_{\alpha+\alpha'}$ *for all* $\alpha, \alpha' \in \Lambda$.
(6) *For all* $\alpha \in \Lambda$, *there exists some* $H_\alpha \in \mathfrak{a}$ *so that* $B(H, H_\alpha) = \alpha(H)$ *for all* $H \in \mathfrak{a}$.

Proof. These statements are stated and proven in Theorem 4.2, Theorem 4.3 in Chapter III of [2] and Proposition 8.3 of [3]. □

3.2. **Cartan decomposition.** The Cartan decomposition is a natural vector space decomposition of any semisimple Lie algebra, and is very useful in the case when the Lie algebra is non-compact. This decomposition is the key to relating Riemannian symmetric spaces of non-compact type and semisimple Lie groups with non-compact Lie algebras. We start this subsection be defining an important concept; the real form of a Lie algebra.

Definition 3.14. Let $\mathfrak{g}$ be a complex Lie algebra and $\mathfrak{h}$ a Lie subalgebra. $\mathfrak{h}$ is a *real form* of $\mathfrak{g}$ if there exists a $\mathbb{C}$ linear isomorphism $\phi : \mathfrak{h}^{\mathbb{C}} \to \mathfrak{g}$ such that $\phi|_{\mathfrak{h}} = \text{id}$.

In the above, $\mathfrak{h}^{\mathbb{C}}$ is the *complexification* of $\mathfrak{h}$, and is defined to be the complexification of $\mathfrak{h}$ as a vector space, equipped with the bracket

$$[X + iY, Z + iW]_{\mathfrak{h}^{\mathbb{C}}} := [X, Z]_{\mathfrak{h}} - [Y, W]_{\mathfrak{h}} + i([Y, Z]_{\mathfrak{h}} + [X, W]_{\mathfrak{h}})$$

for all $X, Y, Z, W \in \mathfrak{h}$. Clearly, if $\mathfrak{h}$ is a real form of $\mathfrak{g}$, then the complex Killing form on $\mathfrak{g}$ restricted to $\mathfrak{h}$ is the real Killing form on $\mathfrak{h}$.

Among the real forms of a complex Lie algebra, the compact ones are of special importance. However, just as in the case of Cartan subalgebras, it is not true that every complex Lie algebra has a compact real form.

Example 3.15. $\mathfrak{b}_n(\mathbb{C})$ is a complex Lie algebra with no compact real form, because none of the matrices in $\mathfrak{b}_n(\mathbb{C})$ are conjugate in $\mathfrak{gl}_n(\mathbb{C})$ to a skew-hermitian matrix.

Fortunately, the semisimple condition again saves the day.

Theorem 3.16.

(1) Every complex semisimple Lie algebra has a real form which is compact.

(2) Let $\mathfrak{g}$ be a real semisimple Lie algebra, let $\mathfrak{u}$ be a compact real form of $\mathfrak{g}^{\mathbb{C}}$ and let $\sigma : \mathfrak{g}^{\mathbb{C}} \to \mathfrak{g}^{\mathbb{C}}$ be complex conjugation with respect to $\mathfrak{g}$. Then there is an isomorphisms $\psi : \mathfrak{g}^{\mathbb{C}} \to \mathfrak{g}^{\mathbb{C}}$ such that $\psi(\mathfrak{u})$ is invariant under σ.

Proof. Proof of (1). Let $\mathfrak{g}$ be a complex Lie algebra, and consider a root space decomposition

$$\mathfrak{g} = \mathfrak{a} \oplus \bigoplus_{\alpha \in \Lambda} \mathfrak{g}_\alpha.$$

For each $\alpha \in \Lambda$, define H_α as in (6) of Theorem 3.13 and for each $\alpha \in \Lambda$, choose $X_\alpha \in \mathfrak{g}_\alpha$ so that $[X_\alpha, X_{-\alpha}] = H_\alpha$. One then uses properties of the root space decomposition to check that

$$\mathfrak{u} := \sum_{\alpha \in \Lambda} \mathbb{R}(iH_\alpha) + \sum_{\alpha \in \Lambda} \mathbb{R}(X_\alpha - X_{-\alpha}) + \sum_{\alpha \in \Lambda} \mathbb{R}(i(X_\alpha + X_{-\alpha}))$$

is a compact real form. See Theorem 6.3 in Chapter III of [2] for details.

Proof of (2). Let τ be complex conjugation with respect to $\mathfrak{u}$. Using the fact that both σ and τ are involutions, one can show that the operator $P := (\sigma\tau)^2$ is diagonalizable with positive eigenvalues. Hence, one can define $\psi := P^{\frac{1}{4}}$ and check that $\psi(\mathfrak{u})$ is invariant under σ. See Theorem 7.1 in Chapter III of [2] for details. □

Using the above theorem, we can define a Cartan decomposition of a semsimple Lie algebra in the following way.

Definition 3.17. Let $\mathfrak{g}$ be a real Lie algebra and σ be complex conjugation of $\mathfrak{g}^{\mathbb{C}}$ with respect to $\mathfrak{g}$. A vector space decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ is a *Cartan decomposition* if there exists a compact real form $\mathfrak{u} \subset \mathfrak{g}^{\mathbb{C}}$ such that $\sigma(\mathfrak{u}) = \mathfrak{u}$, $\mathfrak{k} = \mathfrak{g} \cap \mathfrak{u}$ and $\mathfrak{p} = \mathfrak{g} \cap (i\mathfrak{u})$.

Some properties of Cartan decompositions are recorded as the next proposition.

Proposition 3.18. *Let $\mathfrak{g}$ be a semisimple Lie algebra and let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be a Cartan decomposition. Then*

(1) $[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}$, $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$ and $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$.

(2) $B|_{\mathfrak{k} \times \mathfrak{k}}$ is negative definite, $B|_{\mathfrak{p} \times \mathfrak{p}}$ is positive definite and $B(\mathfrak{k}, \mathfrak{p}) = 0$.

(3) If $\mathfrak{g} = \mathfrak{k}' + \mathfrak{p}'$ is another Cartan decomposition, then there exists some $\psi \in Int(\mathfrak{g})$ such that $\psi(\mathfrak{k}) = \mathfrak{k}'$ and $\psi(\mathfrak{p}) = \mathfrak{p}'$.

(4) $\mathfrak{k}$ is a maximal compact subalgebra of $\mathfrak{g}$.

Proof. Parts (1) and (2) are easy consequences of the definition of the Cartan decomposition and the facts that the Killing form is negative definite on a real compact semisimple Lie algebra and the complex Killing form on a complex Lie algebra restricted to a real form gives the Killing form on the real form.

Proof of (3). Let $\mathfrak{u}$ and $\mathfrak{u}'$ be the compact real forms of $\mathfrak{g}^{\mathbb{C}}$ that correspond to the Cartan decompositions $\mathfrak{k}+\mathfrak{p}$ and $\mathfrak{k}'+\mathfrak{p}'$ respectively. Let τ, τ' and σ be complex conjugation of $\mathfrak{g}^{\mathbb{C}}$ with respect to $\mathfrak{u}$, $\mathfrak{u}'$ and $\mathfrak{g}$ respectively, and let $P := (\tau\tau')^2$. One can verify that P is diagonalizable with positive eigenvalues, and σ commutes with P^t for all positive values of t. This means that P^t leaves $\mathfrak{g}$ invariant, so P^t is a one parameter subgroup of $Aut(\mathfrak{g})$. Since $\mathfrak{g}$ is semisimple, $Int(\mathfrak{g})$ is the identity component of $Aut(\mathfrak{g})$ (see Corollary 6.5 in Chapter II of [2]), which means that P^t lies in $Int(\mathfrak{g})$. Check that $\psi := P^{\frac{1}{4}}$ works.

Proof of (4). Suppose that there is a compact subalgebra $\mathfrak{k}_1$ in $\mathfrak{g}$ that properly contains $\mathfrak{k}$. Then there is some non-zero element $X \in \mathfrak{k}_1 \cap \mathfrak{p}$. Let $\mathfrak{u}$ be the compact real form of $\mathfrak{g}^{\mathbb{C}}$ that corresponds to the Cartan decomposition $\mathfrak{k}+\mathfrak{p}$, and let τ be complex conjugation of $\mathfrak{g}^{\mathbb{C}}$ with respect to $\mathfrak{u}$. Then τ preserves $\mathfrak{g}$, and one can check that the bilinear form B_τ on $\mathfrak{g}^{\mathbb{C}}$ defined by $B_\tau(X,Y) = -B(X,\tau Y)$ is symmetric, positive definite, and its restriction to $\mathfrak{g}\times\mathfrak{g}$ is real valued. On the other, one can also show that for $Y, Z \in \mathfrak{g}$, $B_\tau(ad(X)Y,Z) = B_\tau(Y, ad(X)Z)$. This means that $ad(X)$ is diagonalizable over $\mathbb{R}$, and not all of its eigenvalues are zero because $\mathfrak{g}$ is semisimple. However, this implies that the one parameter subgroup $\gamma_{ad(X)}$ in $Int(\mathfrak{g})$ cannot lie in a compact subgroup, contradiction. □

Observe from the definition of a Cartan decomposition that for a compact semisimple Lie algebra $\mathfrak{g}$, the trivial decomposition with $\mathfrak{k} = \mathfrak{g}$ and $\mathfrak{p} = \{0\}$ is a Cartan decomposition. In fact, (4) of Proposition 3.18 implies that this is the only Cartan decomposition for a compact semisimple Lie algebra. Hence, Cartan decompositions are interesting only in the cases when we are working with a non-compact semisimple Lie algebra.

The next theorem gives an equivalent formulation of the Cartan decomposition, which motivates the following definition.

Theorem 3.19. *Let $\mathfrak{g} = \mathfrak{k}+\mathfrak{p}$ be a vector space decomposition of a semisimple Lie algebra such that $\mathfrak{k} \subset \mathfrak{g}$ is a subalgebra. Then this is a Cartan decomposition if and only if the map $s : \mathfrak{k}+\mathfrak{p} \to \mathfrak{k}+\mathfrak{p}$ defined by $s : (X,Y) \mapsto (X,-Y)$ is a Lie algebra automorphism and has the property that the bilinear form B_s defined by $B_s(X,Y) := -B(X,sY)$ is symmetric and positive definite.*

Proof. For the forward direction, recall that the complex conjugation τ of $\mathfrak{g}^{\mathbb{C}}$ defined in the proof of (4) of Proposition 3.18 preserves $\mathfrak{g}$. This allows one to verify that $s = \tau|_{\mathfrak{g}}$ satisfies the required properties. For the other direction, suppose that such an s exists. It is then an easy exercise to show that $B(\mathfrak{k},\mathfrak{p}) = 0$, $[\mathfrak{k},\mathfrak{p}] \subset \mathfrak{p}$ and $[\mathfrak{p},\mathfrak{p}] \subset \mathfrak{k}$. Using this, one can check that $\mathfrak{u} := \mathfrak{k} + i\mathfrak{p} \subset \mathfrak{g}^{\mathbb{C}}$ is a compact real form of $\mathfrak{g}^{\mathbb{C}}$. The fact that $\mathfrak{g} = \mathfrak{p}+\mathfrak{k}$ is a Cartan decomposition follows from this. □

Definition 3.20. An involutive Lie algebra automorphism $\theta : \mathfrak{g} \to \mathfrak{g}$ is a *Cartan involution* if the bilinear form B_θ defined by $B_\theta(X,Y) := -B(X,\theta Y)$ is symmetric and positive definite.

Example 3.21. Define the vector spaces $\mathfrak{c}_1(F)$ and $\mathfrak{c}_2(F)$ by

$$\mathfrak{c}_1(F) := \{X \in \mathfrak{sl}_n(F) : X = -X^T\} \text{ and } \mathfrak{c}_2(F) := \{X \in \mathfrak{sl}_n(F) : X = X^T\}$$

Observe that we have the vector space decompositions

$$\begin{aligned}\mathfrak{sl}_n(\mathbb{C}) &= \mathfrak{c}_1(\mathbb{C}) \oplus \mathfrak{c}_2(\mathbb{C})\\ \mathfrak{sl}_n(\mathbb{R}) &= \mathfrak{c}_1(\mathbb{R}) \oplus \mathfrak{c}_2(\mathbb{R})\\ \mathfrak{su}(n) &= \mathfrak{c}_1(\mathbb{R}) \oplus i\mathfrak{c}_2(\mathbb{R})\end{aligned}$$

In particular, $\mathfrak{sl}_n(\mathbb{R})$ and $\mathfrak{su}(n)$ are real forms of $\mathfrak{sl}_n(\mathbb{C})$. Since $\mathfrak{su}(n)$ is compact, this also shows that the above decomposition of $\mathfrak{sl}_n(\mathbb{R})$ is a Cartan decomposition, with $\mathfrak{k} = \mathfrak{c}_1(\mathbb{R})$ and $\mathfrak{p} = \mathfrak{c}_2(\mathbb{R})$.

4. Symmetric spaces.

Definition 4.1. A *(Riemannian) symmetric space* M is a connected, simply connected real analytic Riemannian manifold such that for all $p \in M$, there is an involutive isometry $s_p : M \to M$ with the property that $s_p(p) = p$ and $(ds_p)_p(X) = -X$. Also, denote the connected component of the isometry group (equipped with the compact open topology) of M by $I_0(M)$.

We will leave it to the reader to show that M is complete and $I_0(M)$ acts transitively on M.

Theorem 4.2. *$G := I_0(M)$ can be equipped with the structure of a Lie group that acts transitively on M, and $K := Stab_G(p)$ is a compact subgroup.*

Proof. First, topologize G by the compact-open topology. The compactness of K follows from the Riemannian geometry fact that for any connected, complete Riemannian manifold M and any sequence $\{\phi_n\}$ in the isometry group of M, if $\{\phi_n(p)\}$ is uniformly bounded for some $p \in M$, then there is a subsequence of ϕ_n that converges to some isometry ϕ of M.

Since K preserves an inner product on T_pM, it can be realized as a closed subgroup of an orthogonal group of T_pM, and thus has the structure of a Lie group. (Here, we are using the closed subgroup theorem, which states that every closed subgroup of a Lie group is an embedded Lie subgroup.) Next, consider a ball $B_r(p)$ of radius r in M centered at p. If r is sufficiently small, then for every $q \in B_r(p)$, there is a unique geodesic between p and q. Let q' be the midpoint between p and q along this geodesic, and note that $T_q := s_{q'}s_p$ is an isometry in $I_0(M)$ that maps p to q. Hence, $B_r(p)$ can be identified with a subset S of $I_0(M)$, and S inherits an analytic structure from $B_r(p)$. One can then show that $S \times K$ is an open subset of G, and has a natural product analytic structure. Doing this for every point $p \in M$ allows us to build an analytic structure on G, for which multiplication and inversion are real analytic. □

For every $p \in M$, the involution $s_p : T_pM \to T_pM$ induces a group automorphism $\sigma_p : G \to G$ defined by $\sigma_p : g \mapsto s_p g s_p$. This in turn induces the involutive Lie algebra isomorphism $\theta_p := (d\sigma_p)_{\mathrm{id}} : \mathfrak{g} \to \mathfrak{g}$. Since θ_p is an involution, it is diagonalizable, and each of its eigenvalues are either 1 or -1. Hence, we can decompose

$$\mathfrak{g} = \{X \in \mathfrak{g} : \theta_p(X) = X\} + \{X \in \mathfrak{g} : \theta_p(X) = -X\}.$$

Some properties of this decomposition is listed in the next proposition.

Proposition 4.3.

(1) *The subspace $\{X \in \mathfrak{g} : \theta_p X = X\}$ of $\mathfrak{g}$ is the Lie algebra $\mathfrak{k}$ of K.*

(2) *Let* $\mathfrak{p} := \{X \in \mathfrak{g} : \theta_p(X) = -X\}$. *Then* $\mathfrak{p}$ *is* $Ad(K)$*-invariant.*
(3) *Let* $\Pi_p : G \to M$ *be the projection given by* $\Pi_p : g \mapsto g \cdot p$. *Then* $(d\Pi_p)_{\mathrm{id}} : \mathfrak{g} \to T_pM$ *is surjective with kernel* $\mathfrak{k}$.
(4) $\gamma(t)$ *is a geodesic in* M *with* $\gamma(0) = p$ *if and only if* $\gamma(t) = \gamma_X(t) \cdot p$ *for some* $X \in \mathfrak{p}$.

Proof. Observe that if $k \in K$, then $s_pks_p = k$, and if $g \in G \setminus K$, then $s_pgs_p \neq g$. Part (1) follows from this fact. This fact also implies that $C_k\sigma_p(g) = \sigma_pC_k(g)$ for all $k \in K$ and all $g \in G$, which means that $Ad(k)\theta_p(X) = \theta_pAd(k)(X)$ for all $k \in K$ and all $X \in \mathfrak{g}$. In particular, (2) holds.

It is clear that the kernel of $(d\Pi_p)_{\mathrm{id}}$ has kernel $\mathfrak{k}$ because $\Pi_p^{-1}(p) = K$. The surjectivity of Π is a consequence of the following: for arbitrarily small r, the open subset $S \times K$ in G defined in the proof of Theorem 4.2 is mapped by Π_p onto $B_r(p)$. Hence, (3) holds. In particular, (3) implies that $(d\Pi_p)_{\mathrm{id}}|_{\mathfrak{p}} : \mathfrak{p} \to T_pM$ is an isomorphism of vector spaces. Hence, one needs only to verify that $\gamma_X(t) \cdot p$ is indeed a geodesic for every $X \in \mathfrak{p}$. Let $\gamma(t)$ be a geodesic in M with $\gamma(0) = p$ and $\gamma'(0) = (d\Pi_p)_{\mathrm{id}}(X)$. For all $t \in \mathbb{R}$, let p_t be the midpoint between p and $\gamma(t)$ and define $T_{\gamma(t)} := s_{p_t}s_p$. It is clear that $T_{\gamma(t)} \cdot p = \gamma(t)$, and one can check that $T_{\gamma(t)}$ is a one parameter subgroup of G and $\frac{d}{dt}|_{t=0}T_{\gamma(t)} = X$. □

Example 4.4.

(1) $M = \mathbb{R}^n$, $G = \mathbb{R}^n \rtimes SO(n)$, $K = SO(n)$.
(2) $M = S^n$, $G = SO(n+1)$, $K = SO(n)$.
(3) $M = \mathbb{H}^n$, $G = SO^+(n,1)$, $K = SO(n)$.
(4) $M = SL_n(\mathbb{R})/SO(n)$ equipped with metric induced by B_θ, where θ is the Cartan involution corresponding to $\mathfrak{so}(n)$, $G = PSL_n(\mathbb{R})$, $K = PSO(n)$.

4.1. **Classification theorem.** For the rest of this section, we will focus on symmetric spaces M such that the Lie algebra of $I_0(M)$ has the property known as "non-compact type". Before we do that however, we will use this subsection to give a brief description of a well-known classification of symmetric spaces, so as to motivate the non-compact type restriction we impose. Since this subsection is more of an aside, statements will be made without giving any proofs.

Definition 4.5. Let M be a symmetric space.

(1) M is *irreducible* if it cannot be written as a non-trivial product of symmetric spaces.
(2) M is of *non-compact type* if M has no Euclidean factors and non-positive sectional curvature.
(3) M is of *compact type* if M has no Euclidean factors and non-negative sectional curvature.

The next theorem motivates the terminology used in the above theorem. This is a deep classification theorem, and one can find a very detailed treatment of it in Section 1 through Section 4 of Chapter V in [2].

Theorem 4.6. *Let* M *be a Riemannian symmetric space and* $G := I_0(M)$.

(1) M *is of non-compact type if and only if* G *is semisimple and* θ_p *is a Cartan involution for all* $p \in M$. *In particular,* $\mathfrak{g}$ *is not compact.*
(2) M *is of compact type if and only if* G *is semisimple and compact.*

(3) $M = M_0 \times \cdots \times M_k$, *where* M_0 *is trivial or Euclidean and* M_i *is trivial or irreducible of compact or non-compact type for all* $i = 1, \ldots, k$.

Observe that the above theorem implies in particular that any symmetric space can be decomposed into a product according to curvature.

4.2. **Flats and the restricted root space decomposition.** It turns out that the configuration of the collection of totally geodesic Euclidean subspaces in M determines a lot of the geometry of M. We will now study these subspaces, also known as flats.

Definition 4.7.

(1) A *flat* in M is a totally geodesic embedding of Euclidean space in M.
(2) A flat is *maximal* if it is not properly contained in another flat.
(3) The *rank* of M is the dimension of a maximal flat.

In the above definition, the rank of M is well-defined because for any two maximal flats, there is an isometry of M that takes one of these flats to the other. This fact, together with other properties of flats, are stated as the next proposition.

Proposition 4.8. *Let* M *be a symmetric space and let* $G := I_0(M)$.

(1) *Let* $p \in M$ *and let* $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ *be the vector space decomposition corresponding to* p. *The map*

$$\begin{aligned} \{\text{Maximal abelian subalgebras in } \mathfrak{p}\} &\to \{\text{Maximal flats containing } p\} \\ \mathfrak{a} &\mapsto \exp(\mathfrak{a}) \cdot p \end{aligned}$$

is a bijection.

(2) *Every geodesic is contained in a maximal flat.*

(3) *Let* F_1, F_2 *be maximal flats in* M. *Then there is some* $g \in G$ *such that* $g \cdot F_1 = F_2$.

Proof. For symmetric spaces, there is a well-known formula for the curvature tensor in terms of the Lie bracket. More precisely, let $p \in M$ and let $X, Y, Z \in T_pM$. Then the isomorphism $(d\Pi_p)_{\mathrm{id}}$ identifies X, Y, Z with vectors in $\mathfrak{p}$, and $R(X,Y)Z = -[[X,Y],Z]$ (see Theorem 4.2 in Chapter IV of [2] for a proof). Part (1) follows easily from this. Part (2) is a consequence of the fact that G acts transitively on the unit tangent bundle of M.

Proof of (3). By (1) and the transitivity of the action of G on M, it is sufficient to show that if $\mathfrak{k} + \mathfrak{p} = \mathfrak{g}$ is a Cartan decomposition corresponding to some $p \in M$ and for any pair of maximal abelian subalgbras $\mathfrak{a}_1$ and $\mathfrak{a}_2$ in $\mathfrak{p}$, there is some $k \in K$ such that $Ad(k)\mathfrak{a}_1 = \mathfrak{a}_2$. Choose $X_i \in \mathfrak{a}_i$ so that $ad(X_i)$ is diagonalizable with pairwise distinct eigenvalues, and consider the continuous function $f : K \to \mathbb{R}$ given by $f(k) = B(Ad(k)X_1, X_2)$. One can then show that if k is a minimum of f, then $Ad(k)\mathfrak{a}_1 = \mathfrak{a}_2$. See Lemma 6.3 in Chapter V of [2] for details. □

Example 4.9.

(1) $\mathbb{H}^n$ and S^n are rank 1 symmetric spaces.

(2) $SL_n(\mathbb{R})/SO(n)$ has rank $n-1$, and a maximal flat containing the point $\mathrm{id}\cdot SO(n)$ is

$$\left\{\begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{pmatrix}\cdot SO(n) : \prod_{i=1}^{n} a_i = 1\right\}.$$

When M is of non-compact type, the maximal flats in M allow us to construct a vector space decomposition, known as the restricted root space decomposition, of $\mathfrak{g}$. This decomposition is quite similar to the root space decomposition described in Section 3.1, and has the advantage that it can be defined even for real, non-compact semisimple Lie algebras.

To describe this decomposition, we start with a Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ of the Lie algebra of $G = I_0(M)$. One can check that for all $X \in \mathfrak{p}$, $ad(X)$ is self adjoint with respect to the positive definite bilinear form B_θ, where θ is the Cartan involution corresponding to the Cartan decomposition we started with. In particular, every element in $ad(\mathfrak{p})$ is diagonalizable. Hence, if we choose a maximal abelian subalgebra $\mathfrak{a}$ in $\mathfrak{p}$, then $ad(\mathfrak{a})$ is simultaneously diagonalizable, and will play the role of the Cartan subalgebra in the root space decomposition.

As we did in the root space decomposition, we define

$$\mathfrak{g}_\alpha := \{X \in \mathfrak{g} : ad(H)X = \alpha(H)X \text{ for all } H \in \mathfrak{a}\}$$

for any $\alpha \in \mathfrak{a}^*$, and

$$\Lambda := \{\alpha \in \mathfrak{a}^* : \mathfrak{g}_\alpha \neq \{0\}\} \setminus \{0\}.$$

This allows us to write

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha\in\Lambda} \mathfrak{g}_\alpha,$$

which is the *restricted root space decomposition.*

Unlike the root space decomposition, it is not necessarily true that $\mathfrak{g}_0 = \mathfrak{a}$ or that $\mathfrak{g}_\alpha$ is 1-dimensional for all $\alpha \in \Lambda$. However, we do have the following properties.

Proposition 4.10.

(1) If $\alpha \in \Lambda$, then $-\alpha \in \Lambda$ and $\theta : \mathfrak{g}_\alpha \to \mathfrak{g}_{-\alpha}$ is an isomorphism for all $\alpha \in \Lambda$.
(2) If $\theta(\mathfrak{g}_0) = \mathfrak{g}_0$.
(3) For all $\alpha, \alpha' \in \Lambda$, $[\mathfrak{g}_\alpha, \mathfrak{g}_{\alpha'}] \subset \mathfrak{g}_{\alpha+\alpha'}$.

Proof. There are consequences of routine computations. □

4.3. **Boundary of a symmetric space of non-compact type.** By Theorem 4.6, we see that if M is compact type, then it is in particular compact, so it does not admit a natural boundary. Also, if M is Euclidean, then its boundary is easy to understand. Hence in this subsection, we will focus on the case when M is of non-compact type, and describe its boundary and the stabilizers of points its boundary. **For the rest of this section, we will assume that M is of non-compact type.**

Using the restricted root space decomposition, we will describe the stabilizers of points on the boundary of M. But first, we need to define what we mean by boundary.

Definition 4.11. Let M be a symmetric space of non-compact type. The *boundary* of M,

$$\partial M := \{ \text{ directed unit speed geodesics in } M\}/\sim$$

where $\gamma_1 \sim \gamma_2$ if $d_M(\gamma_1(t), \gamma_2(t))$ is uniformly bounded above for $t \geq 0$.

The next proposition holds in any Riemannian metric space of nonpositive curvature. In particular, in holds for our symmetric space of non-compact type, M.

Proposition 4.12. *Choose $p \in M$. Then the map $T^1_p(M) \to \partial M$ which sends each unit tangent vector v to the unique unit speed geodesic γ so that $\gamma(0) = p$ and $\gamma'(0) = v$, is a bijection.*

This proposition allows us to define a topology on ∂M which is independent of the choice of $p \in M$. The action of G on M extends to continuously to an action on ∂M because this action preserves d_M and maps unit speed geodesics to unit speed geodesics.

Definition 4.13. A subgroup $P \leq G$ is *parabolic* if it is the stabilizer of a point in ∂M.

We want to understand the parabolic subgroups of G. The main tool to do so is the following theorem.

Theorem 4.14. *Let $\xi \in \partial M$ and let G_ξ be the corresponding parabolic subgroup. Choose any $p \in M$, let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the Cartan decomposition corresponding to p and let $X \in \mathfrak{p}$ such that $[\gamma_X(t) \cdot p] = \xi$. Then $g \in G_\xi$ if and only if*

$$\lim_{t\to\infty} \gamma_X(t)^{-1} g \gamma_X(t)$$

exists.

Proof. First, observe that $g \in G_\xi$ if and only if

$$d_M(\gamma_X(t) \cdot p, g\gamma_X(t) \cdot p) = d_M(p, \gamma_X(t)^{-1} g\gamma_X(t) \cdot p)$$

is uniformly bounded above for all $t \geq 0$.

Now, suppose that $\phi := \lim_{t\to\infty} \gamma_X(t)^{-1} g\gamma_X(t)$ exists. Then

$$\lim_{t\to\infty} d_M(p, \gamma_X(t)^{-1} g\gamma_X(t) \cdot p) = d_M(p, \phi \cdot p),$$

which implies that $d_M(p, \gamma_X(t)^{-1} g\gamma_X(t) \cdot p)$ is uniformly bounded above for all $t \geq 0$.

On the other hand, if $d_M(p, \gamma_X(t)^{-1} g\gamma_X(t) \cdot p)$ is uniformly bounded above for all $t \geq 0$, then $\gamma_X(t_n)^{-1} g\gamma_X(t_n)$ converges in G up to subsequence because $\Pi_p : G \to M$ has compact fibers. One then uses the fact that $ad(X)$ is symmetric for all $X \in \mathfrak{p}$ to show that the sequence actually converges. See Proposition 2.17.3 in [1] for details. □

With the description of parabolic subgroups given in Theorem 4.14, we can give a more explicit description of what the Lie algebras of parabolic subgroups have to be, up to conjugation.

Corollary 4.15. *Assume the same hypothesis as Theorem 4.14. Also, let $\mathfrak{a} \subset \mathfrak{p}$ be a Cartan subalgebra such that $\gamma_X(t) \cdot p \subset \exp(\mathfrak{a}) \cdot p$. Then the Lie algebra $\mathfrak{g}_\xi$ of G_ξ can be given by*

$$\mathfrak{g}_\xi = \mathfrak{a} \oplus \bigoplus_{\alpha\in\Lambda:\alpha(X)\geq 0} \mathfrak{g}_\alpha$$

Proof. On the level of Lie algebras, Theorem 4.14 says that $Z \in \mathfrak{g}_\xi$ if and only if $\lim_{t\to\infty} Ad(\gamma_X(t)^{-1})Z$ exists. Suppose that $Z \in \mathfrak{g}_\alpha$ for some $\alpha \in \Lambda \cup \{0\}$. Then

$$Ad(\gamma_X(t)^{-1})Z = \exp(-ad(X)t)Z = \exp(-\alpha(X)t)Z,$$

so $\lim_{t\to\infty} Ad(\gamma_X(t)^{-1})Z$ exists if and only if $\alpha(X) \geq 0$. The corollary follows easily from this. □

Example 4.16. Let p be the identity coset of $SL_3(\mathbb{R})/SO(3)$ and consider the vector

$$X = \begin{pmatrix} a_0 & 0 & 0 \\ 0 & b_0 & 0 \\ 0 & 0 & c_0 \end{pmatrix} \in \mathfrak{sl}_3(\mathbb{R}).$$

Observe that

$$\mathfrak{a} := \left\{ \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix} \cdot SO(3) : a_1 a_2 a_3 = 1 \right\}$$

is a flat containing the geodesic $\gamma_X(t) \cdot p$. Let $\xi = [\gamma_X(t) \cdot p] \in \partial M$. By Corollary 4.15, we see that if $a_0 > b_0 > c_0$, then

$$G_\xi = \left\{ \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix} \in SL_3(\mathbb{R}) \right\}.$$

On the other hand, if $a_0 = b_0 > c_0$, then

$$G_\xi = \left\{ \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix} \in SL_3(\mathbb{R}) \right\}.$$

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