EXISTENCE THEORY FOR HARMONIC METRICS

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These are the notes of a talk given by the author in Asheville at the workshop "Higgs bundles and Harmonic maps" in January 2015. It aims to sketch the proof of the famous Corlette-Donaldson Theorem which gives the existence of a harmonic metric on flat G-bundles associated to reductive representation.

Let G be a complex semi-simple Lie group and Σ a genus $g > 1$ closed oriented surface. We define the *character variety* $\mathscr{R}(\Sigma, G)$ as the quotient of reductive homomorphisms $\mathrm{Hom}^{red}(\pi_1(\Sigma), G)$ by the action of G by conjugation. For each complex structure on Σ, the non-abelian Hodge theorem provides a parametrization of the character variety $\mathscr{R}(\Sigma, G)$ by the moduli space of poly-stable G-Higgs bundles.

The proof of the non-abelian Hodge theorem contains two main steps. The first one, often called the Corlette-Donaldson theorem, gives the existence of a harmonic metric in the gauge orbit of the flat G-bundle associated to a (conjugacy class of) representation $\rho : \pi_1(\Sigma) \to G$ as soon as ρ is reductive. The second step, called the Hitchin-Kobayashi correspondence, relates G-Higgs bundles satisfying Hitchin equations to poly-stable G-Higgs bundles.

From now on, we will restrict ourselves to the case $G = SL_n(\mathbb{C})$. The proofs and statements in the general case are very similar. However, the $SL_n(\mathbb{C})$ case allows to work in the category of vector bundles and not in the category of principal G-bundles, and simplifies the objects. People familiar with the theory of principal G-bundles would not have difficulties in translating the statements and proofs in the language of principal G-bundles.

These notes have been written using the following articles and books: [Cor88, ES64, Ham75, LW08] and the very good survey [Wen12].

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1. Moduli space of flat connections

Let $E \longrightarrow \Sigma$ be a rank n complex vector bundle over Σ whose first Chern class is 0 (equivalently, the determinant bundle is topologically trivial).

Definition 1. *A* ***connection*** *on E is a $\mathbb{C}$-linear map*

$$\nabla : \Omega^0(\Sigma, E) \longrightarrow \Omega^1(\Sigma, E),$$

satisfying the Leibniz rule

$$\nabla(fs) = df \otimes s + f\nabla s,$$

where $s \in \Omega^0(\Sigma, E)$ and $f \in \Omega^0(\Sigma)$.

Denote by $\mathscr{C}$ the space of connections on E which induce the trivial one on the determinant bundle (this condition corresponds to representation into $SL_n(\mathbb{C})$). Note that the difference between two connections $\nabla_1, \nabla_2 \in \mathscr{C}$ is tensorial, that is

$$(\nabla_2 - \nabla_1)(fs) = f(\nabla_2 - \nabla_1)(s),$$

for $s \in \Omega^0(\Sigma, E)$ and $f \in \Omega^0(\Sigma)$. In particular, $\nabla_2 - \nabla_1 \in \Omega^1(\Sigma, \mathrm{End}_0(E))$ (where $\mathrm{End}_0(E)$ is the bundle of traceless endomorphisms of E) and so $\mathscr{C}$ is an affine space modelled on $\Omega^1(\Sigma, \mathrm{End}_0(E))$.

Definition 2. *Given a connection $\nabla \in \mathscr{C}$, one can associates its* ***curvature***

$$F_\nabla := \nabla \wedge \nabla \in \Omega^2(\Sigma, End_0(E)).$$

A connection is ***flat*** *when $F_\nabla = 0$.*

Denote by $\mathscr{C}_0 \subset \mathscr{C}$ the subspace of flat connections.

Definition 3. *The* ***gauge group*** *is*

$$\begin{aligned} \mathscr{G} : \ &= \ \{g \in \Omega^0(\Sigma, End(E)),\ \det g = 1\} \\ &= \ \Omega^0(\Sigma, SL_n(\mathbb{C})). \end{aligned}$$

We have an action of $\mathscr{G}$ on $\mathscr{C}$ by conjugation. Note that, one easily checks that for $g \in \mathscr{G}$ and $\nabla \in \mathscr{C}$,

$$F_{g.\nabla} = gF_\nabla g^{-1}.$$

In particular, $\mathscr{G}$ preserves $\mathscr{C}_0$.

Proposition 1. *We have a canonical identification between $Hom(\pi_1(\Sigma), SL_n(\mathbb{C}))$ and $\mathscr{C}_0/\mathscr{G}$.*

Proof. Given $\rho \in \mathrm{Hom}(\pi_1(\Sigma), SL_n(\mathbb{C}))$, one can construct a rank n complex vector bundle $E_\rho \longrightarrow \Sigma$ as follow:

$$E_\rho = \widetilde{\Sigma} \times \mathbb{C}^n / \pi_1(\Sigma),$$

where $\widetilde{\Sigma}$ is the universal cover of Σ and the action of $\pi_1(\Sigma)$ on $\widetilde{\Sigma} \times \mathbb{C}^n$ is given by $\gamma.(x, v) = (\gamma x, \rho(\gamma)v)$ (where γx is the action of $\gamma \in \pi_1(\Sigma)$ by deck transformation on $\widetilde{\Sigma}$). The canonical connection on $\widetilde{\Sigma} \times \mathbb{C}^n$ descends to a flat connection ∇_ρ on E_ρ. In particular, the first Chern class of E_ρ vanishes and E_ρ is homeomorphic to E.

Now, given a flat connection $\nabla \in \mathscr{C}_0$, a closed smooth path $\gamma : [0, 1] \to \Sigma$, and a basis B of $E_{\gamma(0)}$, one can consider the basis B' obtained by parallel transport of B_0 along γ. It follows that B and B' are two basis of the same vector space which differ by a unique $g_{B,\gamma} \in SL_n(\mathbb{C})$. By flatness of ∇, $g_{B,\gamma}$ only depends on the homotopy class of γ in $\pi_1(\Sigma)$. It follows that we can define a map

$$hol_{\nabla,B} : \pi_1(\Sigma) \longrightarrow SL_n(\mathbb{C}),$$

which associates to an homotopy class of closed path γ the element $g_{B,\gamma}$ of the above construction. Note that if B' is another basis, $hol_{\nabla,B'}$ is conjugate to $hol_{\nabla,B}$. So we get the holonomy map

$$hol : \nabla \longrightarrow \mathrm{Hom}\big(\pi_1(\Sigma), SL_n(\mathbb{C})\big)/SL_n(\mathbb{C}).$$

Moreover, we check that two gauge equivalent flat connections give rise to the same class of holonomy representation. □

We finish this section with a definition:

Definition 4. *A flat connection $\nabla \in \mathscr{C}_0$ on E is* ***reductive*** *if any ∇-invariant subbundle has a ∇-invariant complement. A representation $\rho \in Hom\big(\pi_1(\Sigma), SL_n(\mathbb{C})\big)$ is reductive if its associated flat connection is reductive.*

2. Harmonic metrics

Let $\rho : \pi_1(\Sigma) \longrightarrow SL_n(\mathbb{C})$ and $E = E_\rho$.

Definition 5. *Let h be a hermitian metric on E. A connection d_A is* ***unitary*** *if for each sections s_1 and s_2 of E,*

$$d\langle s_1, s_2\rangle_h = \langle d_A s_1, s_2\rangle_h + \langle s_1, d_A s_2\rangle_h,$$

here $\langle ., .\rangle_h$ is the hermitian product h.

In order to define harmonic metrics, we need an interpretation of hermitian metrics in terms of equivariant maps. Set

$$D := SL_n(\mathbb{C})/SU(n) = \{\text{positive hermitian matrices } M \text{ with } \det M = 1\}.$$

We have an action on $SL_n(\mathbb{C})$ on D given by

$$g.M = (g^{-1})^* M g^{-1}.$$

Proposition 2. *A hermitian metric on E is equivalent to a ρ-equivariant map*

$$u : \widetilde{\Sigma} \longrightarrow D,$$

that is a map satisfying $u(\gamma x) = \left(\rho(\gamma)^{-1}\right)^* u(x)\rho(\gamma)^{-1}$.

Proof. Let u be such a map and s be a section of E seen as an ρ-equivariant map $s : \widetilde{\Sigma} \longrightarrow \mathbb{C}^n$. Define

$$\|s\|_u(x) := \langle s(x), u(x)s(x)\rangle_{\mathbb{C}^n}.$$

We easily check that

$$\|s\|_u^2(\gamma x) = \langle \rho(\gamma)s(x), \left(\rho(\gamma)^{-1}\right)^* u(x)s(x)\rangle_{\mathbb{C}^n} = \|s\|_u^2(x).$$

So $\|s\|_u^2$ is ρ-invariant and descends to a function over Σ.

On the other hand, let h be a hermitian metric on E and take s_1 and s_2 two sections of E. Take $x \in \Sigma$, $\widetilde{x} \in \widetilde{\Sigma}$ and choose $\widetilde{s_i} : (\widetilde{\Sigma}) \longrightarrow \mathbb{C}^n$ two ρ-equivariant maps representing s_i. It follows that there exists a $u(\widetilde{x}) \in D$ so that

$$\langle s_1(x), s_2(x)\rangle_h = \langle \widetilde{s_1}(\widetilde{x}), u(\widetilde{x})s_2(\widetilde{x})\rangle_{\mathbb{C}^n}.$$

As the right hand side is ρ-invariant,

$$\langle \widetilde{s_1}(\widetilde{x}), u(\widetilde{x})s_2(\widetilde{x})\rangle_{\mathbb{C}^n} = \langle \rho(\gamma)\widetilde{s_1}(\widetilde{x}), u(\widetilde{\gamma x})\rho(\gamma)s_2(\widetilde{x})\rangle_{\mathbb{C}^n}.$$

It follows that $u(\widetilde{x}) = \rho(\gamma)^* u(\gamma\widetilde{x})\rho(\gamma)$, and so u is ρ-equivariant. □

Note that D carries an invariant metric (the Killing metric), and we denote by ∇ the associated Levi-Civita connection. So, **given a metric** g **on** Σ, one can define the energy of a Hermitian metric $u \in W_\rho^{1,2}(\widetilde{\Sigma}, D)$

(the Sobolev space of ρ-equivariant map admitting a weak differential in L^2) by:

$$\mathscr{E}_\rho(u) := \frac{1}{2}\int_\Sigma \|du\|^2 dv_g.$$

Here, $du \in \Gamma(T^*\widetilde{\Sigma} \otimes u^*TD)$, the norm of du is taken with respect to the product metric and dv_g is the area form associated to g. Note that the integral is well-defined as $\|du\|^2$ is ρ-invariant and so descends to a function on Σ.

Remark 1. *The energy of u only depends on the conformal class of g. So the energy of a hermitian metric u can be defined for each complex structure on Σ.*

Definition 6. *A* ***harmonic metric*** *is a $\mathscr{C}^2$ ρ-equivariant map $u : \widetilde{\Sigma} \longrightarrow D$ which is a critical point of the energy functional.*

We would like a local description of harmonic metrics. Given a ρ-equivariant map u, one can associate its **tension field**

$$\tau(u) := d_\nabla^* du,$$

where d_∇^* is the dual of the covariant derivative of forms with value in u^*TD.

Proposition 3. *Let $\psi := \frac{d}{dt}_{|t=0} u_t$ where $(u_t)_{t\in I}$ is a smooth path of ρ-equivariant maps with $u_0 = u$. We have the following:*

$$\frac{d}{dt}_{|t=0}\mathscr{E}_\rho(u_t) = \int_\Sigma \langle \tau(u), \psi\rangle dv_g,$$

where $\langle .,. \rangle$ is the scalar product with respect to the pull-back by u of the Killing metric on D.

Proof. Note that we have $\frac{d}{dt}_{|t=0} du_t = d_\nabla \psi$. It follows that

$$\begin{aligned} \frac{d}{dt}_{|t=0}\mathscr{E}_\rho(u_t) &= \frac{d}{dt}_{|t=0}\frac{1}{2}\int_M \|du\|^2 dv_g \\ &= \int_\Sigma \langle d_\nabla \psi, du\rangle dv_g \\ &= \int_\Sigma \langle \psi, d_\nabla^* du\rangle dv_g. \end{aligned}$$

□

Corollary 1. *A $\mathscr{C}^2$ ρ-equivariant map u is a harmonic metric if and only if $\tau(u) = 0$.*

We are now ready to state the main theorem:

Theorem (Corlette-Donaldson-Labourie)**.** *The following are equivalent:*

(1) *The $\mathscr{G}$-orbit of a flat connection A admits a harmonic metric.*
(2) *The representation $\rho_A \in Hom(\pi_1\Sigma, G)$ is reductive.*

Note that this theorem provides an identification, when fixing a complex structure on Σ, between the moduli space of flat reductive connections and the moduli space of harmonic bundles.

3. Proof

(1) $\implies$ (2)

Suppose u is a harmonic metric and ρ is not reductive. Let $E_1 \subset E$ be an ∇-invariant sub-bundle and let E_2 its orthogonal complement. Denote by $n_i = \dim E_i$. The flat connection ∇ decomposes as follow (using the metric u):

$$A = \begin{pmatrix} A_1 & \beta \\ 0 & A_2 \end{pmatrix} = d_A + \Psi = \begin{pmatrix} d_{A_1} + \Psi_1 & \beta \\ 0 & d_{A_2} + \Psi_2 \end{pmatrix}$$

where $\beta \in \Omega^0(X, Hom(E_1, E_2))$ is called the second fundamental form, the d_{A_i} are unitary and $\Psi_i \in \Omega^1(X, i\mathfrak{su}(n_i))$. One easily checks that

$$\Psi = \begin{pmatrix} \Psi_1 & \frac{1}{2}\beta \\ \frac{1}{2}\beta^* & \Psi_2 \end{pmatrix},$$

and that

$$\mathscr{E}_\rho(u) = \|\Psi\|^2 = \|\Psi_1\|^2 + \|\Psi_2\|^2 + \|\beta\|^2.$$

Let $\xi = (-n_2)_{|E_1} \oplus n_{1|E_2} \in Lie(\mathscr{G})$. Let

$$u_t := \exp(t\xi)u.$$

We obtain

$$\|\Psi_t\|^2 = \|\Psi_1\|^2 + \|\Psi_2\|^2 + e^{-tn/2}\|\beta\|^2.$$

But, as u is harmonic, $\frac{d}{dt}_{|t=0}\|\Psi_t\| = 0$, so $\beta = 0$ which gives a contradiction.

(2) $\implies$ (1)

3.1. **Definitions and notations.** Let (M, g) be a Riemannian manifold and $E \longrightarrow (M, g)$ be a complex vector bundle equipped with a hermitian metric. For $\eta, \xi \in \Gamma(E)$, one defines the scalar product

$$\langle \eta, \xi \rangle_E := \int_M (\eta, \overline{\xi}) dv_g.$$

Recall that the metric on E induces a covariant derivatives on forms with value in E that we denote

$$d_\nabla : \Gamma(E) \longrightarrow \Gamma(T^*M \otimes E).$$

For $k \in \mathbb{N} \cup \{\infty\}$ and $\alpha \in (0, 1)$, we define the following vector spaces:

- $\mathscr{C}^k(E) = \{$sections of E which are $\mathscr{C}^k\}$
- $\mathscr{C}^k_0(E) = \{\eta \in \mathscr{C}^k(E),$ so that η has compact support$\}$
- $\mathscr{C}^{k,\alpha}(E) = \{\eta \in \mathscr{C}^k(E),\ d^k_\nabla \eta \in \mathscr{C}^{0,\alpha}((T^*M)^{\otimes k} \otimes E)\}$
- $L^2(E) = \{\eta \in \Gamma(E),\ \|\eta\|^2_E := \langle \eta, \overline{\eta} \rangle_E < +\infty\}$.
- $L^p(E) := \{\eta \in \Gamma(E), \int_M \|\eta\|^p < +\infty\}$
- $W^{k,p}(E) := \{\eta \in L^p(E),\ \forall i = 1, ..., k,\ d^i_\nabla \eta \in L^p((T^*M)^{\otimes i} \otimes E)\}$
- $W^{k,p}_{loc} := \{\eta \in \Gamma(E),\ \forall K \subset M$ compact $, \eta_{|K} \in W^{k,p}(E_{|K})\}$.

Remark 2. *When E is the trivial vector bundle, we denote these spaces by $L^p(M)$... They identify with spaces of functions over M.*

Given $\eta \in L^p(E)$, we define:

$$\|\eta\|_{L^p} := \left(\int_M \|\eta\|^p dv_g\right)^{1/p},$$

and for $\eta \in W^{k,p}(E)$,

$$\|\eta\|^2_{W^{k,p}} := \sum_{i=0}^{k} \|d^i_\nabla \eta\|^2_{L^p}.$$

With these norms, $L^p(E)$ and $W^{k,p}(E)$ are Banach spaces. Note that in the definition of $W^{k,p}(E)$, we only assume that the i-th covariant derivative exists in a weak sense (that is in the sense of distributions). We recall the Sobolev embedding Theorem

Proposition 4. *(Sobolev Embedding Theorem)* $W^{j+k,p}_{loc}(E) \subset \mathscr{C}^{j,\alpha}(E)$ *for all* $\alpha \in \left(0, k - \frac{m}{p}\right)$ *(where* $m = \dim M$*).*

Also, we recall the multiplication law for Sobolev spaces:

Proposition 5. *(Multiplication Law) If* $\frac{k}{m} - \frac{1}{p} > 0$*, then* $W^{k,p}_{loc}(M)$ *is a Banach algebra (that is, the multiplication of functions in* $W^{k,p}_{loc}(M)$ *are in* $W^{k,p}_{loc}(M)$*).*

3.2. **Gradient flow for harmonic maps.** Let $u : \widetilde{\Sigma} \longrightarrow D$ be a ρ-equivariant map so that $du \in L^2(T^*\widetilde{\Sigma} \otimes u^*TD)$ (we say that $u \in W^{1,2}_\rho(\widetilde{\Sigma}, D)$). We define the following equation for $u : M \times I \longrightarrow D$:

$$\begin{cases} \partial_t u_t &= -\tau(u_t) \\ u(.,0) &= u \end{cases}$$

where $u_t = u(.,t)$.

Note that, if a solution $u(.,t)$ exists for some t, then it is also ρ-equivariant.

In a coordinates system, this equation looks like:

$$\begin{cases} (\partial_t - \Delta)u^a_t &= \Gamma^a_{bc}(u_t)\nabla u^b_t \nabla u^c_t \\ u(.,0) &= u, \end{cases} \tag{1}$$

where Γ^a_{bc} are the Christoffel symbols of D and Δ is the Laplace-Beltrami operator on (M, g).

Short time existence.

Short time existence for the gradient flow of harmonic maps with boundary has been proved by R. Hamilton [Ham75]. The proof is based on a Implicit Functions Theorem. It consists in proving that the equation

$$(\partial_t - \Delta + a\nabla + b)f = g, \tag{2}$$

where a and b are smooth and $g \in L^p(\Sigma \times [0, t_0])$ (for some $t_0 > 0$) always admits a unique solution $f \in W^{2,p}(\Sigma \times [0, t_0])$ (with good boundary conditions). In other words, the operator

$$\begin{array}{rccc} L: & W^{2,p}(\Sigma \times [0,t_0]) & \longrightarrow & L^p(\Sigma \times [0,t_0]) \\ & f & \longmapsto & (\partial_t - \Delta)f + b\nabla f + cf \end{array}$$

is an isomorphism.

We look for a solution of equation (1) of the form $u = u_b + v$ where u_b is a fixed smooth function satisfying the boundary conditions and $v \in W^{2,p}(\Sigma \times [0, t_0])$. Let $P : W^{2,p}(\Sigma \times [0, t_0]) \longrightarrow L^p(\Sigma \times [0, t_0])$ the operator defined by

$$P(v) = \partial_t(u_b + v) + \tau(u_b + v).$$

The differential of P at 0 has the form of equation (2) and so is an isomorphism. By the Implicit Function Theorem for Banach spaces, P maps a neighborhood U of 0 in $W^{2,p}(\Sigma \times [0, t_0])$ to a neighborhood V of $P(u_b)$ in $L^p(\Sigma \times [0, t_0])$.

For $\epsilon > 0$ small enough, the function f equal to 0 for $t \in [0, \epsilon)$ and equal to $P(u_b)$ for $t \in [\epsilon, t_0]$ will be in V. It follows that there exists a function $v \in U$ so that $P(u_b + v) = f$. It means that $u_b + v$ will be solution of equation (1) for $t \in [0, \epsilon)$.

Remark 3. *As it is often the case for non-linear parabolic equations, we can prove in this case that if the maximal time existence T_{max} for a solution to equation (1) is finite, then*

$$\lim_{t \to T_{max}} \|\nabla u_t\| = +\infty.$$

Long time existence

To prove the long time existence for the gradient flow, we only need to get a uniform bound on $\|\nabla u_t\|$ (it is a consequence of Remark 3) where u_t is a solution of equation (1). At this point, the curvature of the target space plays an important role. In fact, the energy density $e_t = \frac{1}{2}\|du_t\|^2$ satisfies the so-called Bochner-Eells-Sampson formula (see [ES64]):

$$(\partial_t - \Delta)e_t = -\|\nabla du_t\|^2 - Ric_X(du_t, du_t) + R_D(du_t, du_t, du_t, du_t),$$

where Ric_X is the Ricci curvature tensor of X and R_D is the Riemann curvature tensor of D. As D is a symmetric space of non-compact type, $R_D \leq 0$. Moreover Ric_X is bounded from below. It follows that e_t satisfies

$$(\partial_t - \Delta)e_t \leqslant Ce_t,$$

for some $C > 0$. We say that e_t is a subsolution of the heat equation.

Now, we use the classical Moser's Harnack inequality for subsolutions of the heat equation:

Proposition 6. *(Moser's Harnack inequality) Let (M, g) a Riemannian manifold and $v : M \times [0, T] \longrightarrow \mathbb{R}$ be a non-negative function. If there exist $(x_0, t_0) \in M \times [0, T]$ and $R > 0$ so that for all $(x, t) \in B(x_0, R) \times [t_0 - R^2, t_0]$ (where $B(x_0, R)$ is the radius R ball centred at x_0) we have*

$$(\partial_t - \Delta)v \leqslant Cv, \text{ for } C > 0,$$

then there exists a $C' > 0$ so that

$$v(x_0, t_0) \leqslant C' R^{-(m+2)} \int_{s=t_0-R^2}^{t_0} \int_{B(x_0,R)} v(x, s) dv_g ds.$$

Here $m = \dim M$.

Applying this to $e(x_0,t_0)$ for $(x_0,t_0)\in\Sigma\times[1,T_max)$ and $R<1$, we get

$$\begin{aligned} e_t(x_0,t_0) &\leqslant C'R^{-4}\int_{s=t_0-R^2}^{t_0}\int_{B(x_0,R^2)} e(x,s)dv_g ds \\ &\leqslant C'R^{-4}\int_{s=t_0-R^2}^{t_0}\mathscr{E}(u_s)ds \end{aligned}$$

But, note that, as u satisfies equation (1), we have

$$\frac{d}{dt}\mathscr{E}(u_t) = -\|\tau(u_t)\|^2,$$

and so

$$\mathscr{E}(u_t)=\mathscr{E}(u_0)-\int_{s=0}^{t}\|\tau(u_s)\|^2ds.$$

In particular $\mathscr{E}(u_t)\leqslant\mathscr{E}(u_0)$ and

$$e_t(x_0,t_0)\leq C'R^{-2}\mathscr{E}(u_0).$$

For $t\in[0,1]$, consider the function

$$a(.,t)=\exp(-Ct)e(.,t).$$

Such a function satisfies

$$(\partial_t-\Delta)a\leq 0.$$

Using the maximum principle, we get that $a(x,t)\leq\max_{x\in\Sigma}a(x,0)$, and so

$$e(x,t)\leq\|u_0\|e^{Ct}\leqslant\|u_0\|e^{C}.$$

In particular, the norm of the gradient of u_t is uniformly bounded and solution to the gradient flow exists to all time.

Convergence to a solution

As $\mathscr{E}(u_t)=\mathscr{E}(u_0)-\int_0^l\|\tau(u_s)\|^2ds\leq 0$, there exists an unbounded increasing sequence $(n_i)_{i\in\mathbb{N}}\subset\mathbb{R}_{>0}$ so that the sequence $(u_i)_{i\in\mathbb{N}}$ where $u_i:=u_{n_i}$ satisfies

$$\tau(u_i)\xrightarrow{L^2}0.$$

It follows that $(u_i)_{i\in\mathbb{N}}$ is a sequence of ρ-equivariant Lipschitz map with uniformly bounded Lipschitz constant.

Proposition 7. *If ρ is irreducible, then $u_i\xrightarrow{\mathscr{C}^{0,1}}u_\infty$ where u_∞ is ρ-equivariant.*

Proof. We claim that as ρ is irreducible, u_i is bounded. In fact, let $p\in\widetilde{\Sigma}$ and suppose that $h_i:=u_i(p)$ is not bounded (see h_i as a determinant one matrix). Choose a sequence $(\epsilon_i)_{i\in\mathbb{N}}\subset\mathbb{R}_{>0}$ so that $\epsilon_ih_i\longrightarrow h_\infty\neq 0$. Let $V:=\mathrm{Ker}(h_\infty)$. Note that V is a proper subspace of $\mathbb{C}^n$ (because $V\neq\mathbb{C}^n$ as $h_\infty\neq 0$ and $V\neq 0$ as $\det h_\infty=0$).

We claim that V is stable by $\rho(\pi_1(\Sigma))$. In fact, let $g^{-1}:=\rho(\gamma)$, and $v\in V$. As the u_i are ρ-equivariant, $d(u_i(p),u_i(p)g^{-1})$ is uniformly bounded as so is $|\langle h_iv,w\rangle-\langle h_igv,gw\rangle|$ for all $w\in\mathbb{C}^n$. It follows that

$$|\langle\epsilon_ih_iv,w\rangle-\langle\epsilon_ih_igv,gw\rangle|\longrightarrow 0.$$

As $v\in V$, we get that $\langle h_\infty gv,gw\rangle=0$ for all $w\in\mathbb{C}^n$ and so $gv\in V$. ☐

It follows that $(u_i)_{i\in\mathbb{N}}$ converges to a weak solution of the harmonic equation.

We want to prove that u_∞ is a strong solution. We have that u_∞ satisfies (in the weak sense)

$$\Delta u^a_\infty + \Gamma^a_{bc}(u_\infty)\nabla u_\infty u^b \nabla u^c_\infty = 0.$$

As $u_i \xrightarrow{\mathscr{C}^{0,1}} u_\infty$, then $u_i \xrightarrow{W^{1,p}_{loc}} u_\infty$ for all $p > 1$. Hence $\nabla u^b_\infty, \nabla u^c_\infty \in L^p_{loc}(\widetilde{\Sigma})$ and so $\nabla u^b_\infty \nabla u^c_\infty \in L^{p/2}_{loc}(\widetilde{\Sigma})$. It follows that $\Delta u^a_\infty \in L^{p/2}_{loc}(\widetilde{\Sigma})$ and so, by Schauder estimates,

$$u^a_\infty \in W^{2,p/2}_{loc}(\widetilde{\Sigma}) \text{ for all } p > 1.$$

As $u^a_\infty \in W^{2,p}_{loc}(\widetilde{\Sigma})$ then $\nabla u^b_\infty, \nabla u^c_\infty \in W^{1,p}_{loc}(\widetilde{\Sigma})$. For $p > 0$, $\frac{1}{2} - \frac{1}{p} > 0$, the multiplication law implies that $\nabla u^b_\infty \nabla u^c_\infty \in W^{1,p}_{loc}(\widetilde{\Sigma})$ and so $u^a_\infty \in W^{3,p}_{loc}(\widetilde{\Sigma})$.

Finally, by Sobolev Embedding Theorem, we get that $u^a_\infty \in \mathscr{C}^2(\widetilde{\Sigma})$ and so u_∞ is a harmonic metric.

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