HARMONIC MAPS TO $\mathbb{R}$-TREES AND MORGAN-SHALEN COMPACTIFICATION

SEMIN KIM

1. Introduction

Here is our plan for this article.

(1) In Section 2, we define the notion of harmonic maps and quadratic differentials. Then we give a harmonic map proof of Teichmuller's theorem by Wolf. (See [2], [9])
(2) In Section 3, we explain a compactification of Teichmuller spaces by Wolf using harmonic maps. (See [2], [4], [10], [11])
(3) In Section 4, we review the Morgan-Shalen compactification and the Korevaar-Schoen limit. Then we give a generalization of Section 3 based on the work of Daskalopoulos, Dostoglou, and Wentworth. (See [1], [2], [7])

2. Harmonic maps and Teichmuller's theorem

In this section, we briefly review Teichmuller's theorem, which states that the Teichmuller space of a compact Riemann surface of genus $g > 1$ is homeomorphic to $\mathbb{R}^{6g-6}$. Our goal is to give a sketch of the harmonic map proof by Wolf [10].

2.1. **Harmonic maps and Hopf differentials.** Let $u : (X, \sigma) \to (Y, \rho)$ a smooth map between Riemann surfaces of genus $g > 1$. Define an energy of the map by

$$E(u) = \int_X |du|^2 dvol.$$

The energy is conformally invariant, so the energy is well-defined on a Riemann surface.

A map u is said to be *harmonic* if u is a critical point of the energy among $C^1(X, Y)$, and Euler-Langrange equation for the energy, or harmonic map equation, is given by

$$u_{z\bar{z}} + (\log \rho)_u u_z u_{\bar{z}} = 0.$$

Date: Dec 30, 2014.

Theorem 2.1. *(Existence, Eells-Sampson [3])*
Let M, N be a compact Riemannian manifolds and N has a nonpositive sectional curvature. Then given a continuous map $f : M \to N$, there exists a harmonic map homotopic to f.

Theorem 2.2. *(Uniquness, Hartman [5])*
Let M, N be a compact Riemannian manifolds and N has a nonpositive sectional curvature. If f_0 and f_1 are homotopic harmonic maps such that $f_t(x)$ is geodesic, then

(i) $E(f_0) = E(f_1) = E(f_t)$.
(ii) the length of geodesic $f_t(x)$ is independent on x.

If N has a negative sectional curvature, then f is unique in its homotopy class or f maps onto a geodesic.

Let (S, σ) be a compact Riemann surface. A *quadratic differential* is a section of $T^*X^{1,0} \otimes T^*X^{1,0}$ and denote a set of all holomorphic sections by $QD(\sigma)$.

Given a map $u : (S, \sigma) \to (T, \rho)$, we associate a quadratic differential, called *Hopf differential*, by

$$\Phi_u := (u^*\rho)^{2,0}.$$

We note that Φ_u is holomorphic if and only if u is harmonic.

2.2. **Teichmuller's theorem.** The *Teichmuller space* $\mathcal{T}(S)$ is defined by a collection of homeomorphisms $\{S \xrightarrow{f} X\}$ up to biholomorphism connected to the identity. Equivalently,

$$\mathcal{T}_{hyp}(S) = \mathrm{Met}_{hyp}(S)/\mathrm{Diff}_0(S),$$

where $\mathrm{Met}_{hyp}(S)$ is a set of all smooth metric with constant curvature -1, $\mathrm{Diff}_0(S)$ is diffeomorphisms isotopic to identity, and $\mathrm{Diff}_0(S)$ acts on $\mathrm{Met}_{hyp}(S)$ by pullback.

We now give a proof of Teichumuller's theorem using harmonic maps by Wolf [10].

Theorem 2.3. *$\mathcal{T}_{hyp}(S)$ is homeomorphic to $\mathbb{R}^{6g-6}$.*

Sketch of the proof. We fix $(S, \sigma) \in \mathcal{T}_{hyp}(S)$ as a base point of $\mathcal{T}_{hyp}(S)$. Given $(S, \rho) \in \mathcal{T}_{hyp}(S)$, there exists a unique harmonic map $u_\rho : (S, \sigma) \to (S, \rho)$ by Theorem 2.1 and $Hopf(u_\rho)$ is a holomorphic quadratic differential. Now consider a map

$$\begin{aligned} \mathcal{H} : \quad \mathcal{T}_{hyp}(S) &\to QD(\sigma) \\ \rho \quad &\mapsto \Phi_{u_\rho} \end{aligned}$$

where $\Phi_{u_\rho} = Hopf(u_\rho)$. Wolf showed that $\mathcal{H}$ is a homeomorphism as follows:

(1) $\mathcal{T}_{hyp}(S)$ is $6g-6$ dimensional manifold by considering a slice of the action of $\text{Diff}_0(S)$.
(2) $QD(\sigma)$ is a vector space and dimension is $6g-6$ by the Riemann-Roch theorem.
(3) $\mathcal{H}$ is well-defined by Hartman's uniqueness theorem.
(4) $\mathcal{H}$ is 1-1 by the Bochner formula.
(5) $\mathcal{H}$ is smooth.
(6) $\mathcal{H}$ is proper because of the properness of the energy $E(u_\rho)$, and the estimate

$$a\int_S |\Phi_{u_\rho}| + b \leq E(u_\rho) \leq c\int_S |\Phi_{u_\rho}| + d.$$

(7) $\mathcal{H}$ is a homeomorphism from the invariance of domain together with the fact that $\mathcal{H}$ is a injective, proper map between same dimensional manifolds.

□

3. Compactification of Teichmuller spaces

In previous section, we developed a parametrization of the Teichmuller space by quadratic differentials. We use this fact to get a compactification of the Teichmuller space by Wolf [10].

3.1. **$\mathbb{R}$-trees and measured foliations.** An $\mathbb{R}$-*tree* is a metric space with a property that any two points are joined by a unique arc which is isometric to an interval in $\mathbb{R}$.

Example 3.1. (1) A simplicial tree, not necessarily locally finite, is an $\mathbb{R}$-tree.
(2) Define a metric on $\mathbb{R}^2$ by $d(p,q) = |p-q|$ if p and q lies on the same vertical line, and $d(p,q) = d(p,p_x)+d(p_x,q_x)+d(q_x,q)$ otherwise, where p_x, q_x are projections to x-axis. $(\mathbb{R}^2, d)$ is an example of non-simplicial $\mathbb{R}$-tree.

A *measured foliation* (F,μ) on a Riemann surface is a singular foliation F with a transverse measure μ. Recall that a *transverse measure* is a map from a smooth transverse arc to F to $\mathbb{R} \geq 0$ which is invariant under leaf-preserving isotopy, and locally induced by $|dy|$ on $\mathbb{R}^2$. We denote the set of measured foliation up to equivalence by $MF(\sigma)$, and $PMF(\sigma)$ for projective version.

A quadratic differential Φ defines a measured foliation in the following way: In a natural coordinate away from zeros, $\Phi(\zeta) = d\zeta^2$. Then the local foliaitons $(\{\text{Re}\zeta = const\}, |d\text{Re}\zeta|)$ patches together to give a measured foliation known as *a vertical foliation.*

We note the correspondence between measured foliations and quadratic differentials:

Theorem 3.2. *(Hubbard-Masur [6])*
Given a measured foliation (F, μ) on a compact Riemann surface S of genus > 1, there exists a unique quadratic differential Φ on S such that a vertical foliation of Φ is equivalent to (F, μ).

3.2. **Thurston and Wolf's compactification.**

Theorem 3.3. *(Thurston)*
$MF(\sigma)$ is homeomorphic to a $6g-6$ dimensional ball and $PMF(\sigma)$ is homeomorphic to a $6g-7$ dimensional sphere.

Thurston gave a compactification $\overline{\mathcal{T}}^{Th}$ by gluing the projective measured foliation to $\mathcal{T}$. Thurston's compactification does not depend on a base point and the action of Γ extends to $\overline{\mathcal{T}}^{Th}$ continuously. Explicitly, Thurston's compactification is defined as follows:

$$\overline{\mathcal{T}}^{Th} = \mathcal{T}(S) \cup PMF = \pi \circ l(\mathcal{T}) \cup \pi \circ I(MF) \subset \mathbb{P}(\mathbb{R}_+^C)$$

for a generating set C, a properly defined embedding $\pi \circ l : \mathcal{T} \to \mathbb{P}(\mathbb{R}_+^C)$ and $\pi \circ I : MF(\sigma) \to \mathbb{P}(\mathbb{R}_+^C)$, and the induced topology from $\mathbb{P}(\mathbb{R}_+^C)$.

We now explain the compactification of the Teichmuller space using harmonic maps by Wolf. Let $\mathcal{T}(\sigma)$ be the Teichmuller space of (S, σ), and $QD(\sigma)$ a set of holomorphic quadratic differentials on a Riemann surface (S, σ). Define a norm of $\Phi \in QD(\sigma)$ by

$$\|\Phi\| = \int_S |\Phi(z)| dvol_S.$$

Let

$$BQD_\sigma = \{\Phi \in QD(\sigma) : \|\Phi\| < 1\},$$
$$SQD_\sigma = \{\Phi \in QD(\sigma) : \|\Phi\| = 1\},$$
$$\overline{BQD_\sigma} = BQD_\sigma \cup SQD_\sigma.$$

Then consider a map

$$\bar{\mathcal{H}} : \mathcal{T}(\sigma) \to BQD_\sigma$$
$$\rho \mapsto \frac{4\mathcal{H}(\rho)}{1 + 4\|\mathcal{H}(\rho)\|}.$$

Since $\bar{\mathcal{H}}$ is a homeomorphism onto its image BQD_σ, we identify $\mathcal{T}(\sigma)$ with BQD_σ, and define a compactification $\overline{\mathcal{T}(\sigma)}^W$ by a compactification on its image:

$$\overline{\mathcal{T}(\sigma)}^W = \mathcal{T}(\sigma) \cup SQD_\sigma = \overline{BQD_\sigma}.$$

Theorem 3.4. *(Wolf [10])*
$\overline{\mathcal{T}}^{Th}$ *and* $\overline{\mathcal{T}(\sigma)}^{W}$ *are homeomorphic. Furthermore,* $\overline{\mathcal{T}(\sigma)}^{W} = \overline{\mathcal{T}}^{W}$ *does not depend on a choice of base point* σ*, and* Γ*-action on* $\mathcal{T}$ *extends continuously to* $\overline{\mathcal{T}}^{W}$*.*

4. Compactification of $SL(2,\mathbb{C})$ character varieties

In this section, we generalize the result of previous section to the case of $SL(2,\mathbb{C})$-character varieties by Daskalopoulos, Dostoglou, and Wentworth [1].

4.1. **The Morgan-Shalen compactification.** Let Γ be a finitely generated group and $\chi(\Gamma) = Hom(\Gamma, SL(2,\mathbb{C}))/\!\!/SL(2,\mathbb{C})$. Let C be a conjugacy classes of Γ and $\mathbb{P}(C) = \mathbb{P}(\mathbb{R}^C)$. Define

$$i : \chi(\Gamma) \to \mathbb{P}(C)$$

by $i(\rho)(\gamma) = \log(|\mathrm{Tr}\rho(\gamma)| + 2)$. Then the Morgan-Shalen compactification is defined by the closure of the image of i in $\mathbb{P}(C)$. Morgan and Shalen proved that $\overline{\chi(\Gamma)}$ is compact and boundary points are *projective length functions* of Γ on an $\mathbb{R}$-tree T. We can restate as follows:

Theorem 4.1. *If* $\rho_k \in \chi(\Gamma)$ *is unbounded, then there exist constants* $\lambda_k \to \infty$ *such that the rescaled length functions* $\frac{1}{\lambda_k} l_{\rho_k}$ *converges to* l_ρ *for* $\rho : \Gamma \to Isom(T)$ *for an* $\mathbb{R}$*-tree* T*.*

4.2. **The Korevaar-Schoen limit.** Let Ω be a set and $f : \Omega \to N$ be a map to a simply connected NPC space (N, d_N). We enlarge a domain to get some convexity as follows.

$$\begin{aligned} \Omega_0 &= \Omega, \\ \Omega_{k+1} &= \Omega_k \times \Omega_k \times [0,1], \\ \Omega_\infty &= \textstyle\bigsqcup_{k=0}^{\infty} \Omega_k/\sim, \end{aligned}$$

where $\sim$ is given by $\Omega_k \hookrightarrow \Omega_{k+1}$, $x \mapsto (x, x, 0)$.
$f_k : \Omega_k \to N$ extends to $f_{k+1} : \Omega_{k+1} \to N$ by linear extension:

$$f_{k+1}(x, y, \lambda) = (1 - \lambda) f_k(x) + \lambda f_k(y).$$

We have a induced map $f_\infty : \Omega_\infty \to N$ and a pullback metric $d_\infty = f_\infty^* d_N$. Define

$$(Z, d_Z) := \overline{(\Omega_\infty/d_\infty, d_\infty)}.$$

Then Z is an NPC space and isometric to the closed convex hull $C(f(\Omega))$.

Now consider a sequence of maps $f_k : \Omega \to (N_k, d_k)$. We say f_k converges to $f : \Omega \to (N_\infty, d_\infty)$ in the *pullback sense*, or *the Korevaar-Schoen sense*, if

(*i*) $d_{k,\infty}$ converges locally uniformly to d_∞.

(*ii*) $f : \Omega \xrightarrow{i} \Omega_\infty \xrightarrow{q} (\overline{\Omega_\infty/d_\infty}, d_\infty)$ and $N_\infty := \overline{\Omega_\infty/d_\infty}$.

The following proposition and theorem are properties and conditions for a convergence in the Korevaar-Schoen sense [7].

Proposition 4.2. *The pullback convergence has the following property.*

(*i*) *If N_k are NPC, then so is N_∞.*
(*ii*) *If f_k are energy minimizer, then so is f.*
(*iii*) *If f_k are Γ-equivariant, then so is f.*

Theorem 4.3. *If $f_k : X \to (N_k, d_k)$ has a uniform modulus of continuity:*

$$d_k(f_k(x), f_k(y)) < C(x)d_X(x, y),$$

then f_k converges in the Korevaar-Schoen sense to $f : X \to N_\infty$.

4.3. **The Daskalopoulos-Dostoglou-Wentworth compactification.**

Theorem 4.4. *(Daskalopoulos, Dostoglou, and Wentworth [1])*

Let X be a compact Riemann surface with genus > 1. Given an unbounded sequence of irreducible $SL(2,\mathbb{C})$-representations ρ_k, corresponding harmonic maps $u_k : \tilde{X} \to \mathbb{H}^3$ converges to u_∞ in the Korevaar-Schoen sense after rescaling, where $u_\infty : \tilde{X} \to T$ for an $\mathbb{R}$-tree T.

Sketch of the proof of Theorem 4.4. Given a sequence of irreducible $SL(2,\mathbb{C})$-representations $\rho_k : \pi_1(X) \to SL(2,\mathbb{C})$, the Donaldson-Corlette theorem provides corresponding harmonic maps $u_k : \tilde{X} \to \mathbb{H}^3$ ($SL(2,\mathbb{C})/SU(2) \cong \mathbb{H}^3$). Harmonic maps u_k satisfies the following estimate (see [9])

$$\sup_{y \in B_R(x)} |du_k|(y) \le C(x, R)[E(u_k)]^{\frac{1}{2}}.$$

If we rescale the metrics $d_{\mathbb{H}^3}$ by $\lambda_k = E(u_k)^{\frac{1}{2}}$ and denote the rescaled maps by $\hat{u}_k : \tilde{X} \to (\mathbb{H}^3, \frac{1}{\lambda_k} d_{\mathbb{H}^3})$, then

$$\sup_{y \in B_R(x)} |d\hat{u}_k|(y) \le C(x, R).$$

Therefore, by Theorem 4.3 $\hat{u}_k$ converges in the Korevaar-Schoen sense to

$$u : \tilde{X} \to N_\infty,$$

where $N_\infty = \overline{\Omega_\infty/d_\infty}$ is the Korevaar-Schoen limit. Then N_∞ is indeed an $\mathbb{R}$-tree because $(\mathbb{H}^3, \frac{1}{\lambda_k} d^3_{\mathbb{H}})$ is $\frac{\delta}{\lambda_k}$-hyperbolic, so N_∞ is 0-hyperbolic NPC space, which is an $\mathbb{R}$-tree. □

Recall that given an action of Γ on $\mathbb{H}^3$ by $\rho : \Gamma \to Isom(\mathbb{H}^3)$, the length function $l_\rho : \Gamma \to \mathbb{R} \ge 0$ is defined by

$$l_\rho(\gamma) = \inf_{x \in \mathbb{H}^3} d_{\mathbb{H}^3}(x, \rho(\gamma)x)$$

for $\gamma \in \Gamma$. Similarly, given a Γ-equivariant map $u : \tilde{X} \to \mathbb{H}^3$, we define the length function of u by

$$l_u(\gamma) = \inf_{x \in \tilde{X}_\infty} d_{\mathbb{H}^3}(u_\infty(x), u_\infty(\gamma x)).$$

Theorem 4.5. *(Daskalopoulos-Dostoglou-Wentworth [1])*
The length function l_u of the action of Γ on T is in the same projective class of the Morgan-Shalen limit of ρ_k. Explicitly, for $\gamma \in \Gamma$,

$$l_u(\gamma) = \lim_k \frac{1}{\lambda_k} l_{\hat{u}_k}(\gamma) = \lim_k \frac{1}{\lambda_k} l_{\rho_k}(\gamma) = l_\rho(\gamma).$$

References

[1] G. Daskalopoulos, S. Dostoglou, and R. Wentworth. Character varieties and harmonic maps to **R**-trees. *Math. Res. Lett.*, 5(4):523–533, 1998.
[2] G. Daskalopoulos and R. Wentworth. Harmonic maps and teichmüller theory. *Handbook of Teichmüller theory*, 1:33–109, 2007.
[3] J. Eells and J. H. Sampson. Harmonic mappings of riemannian manifolds. *American Journal of Mathematics*, pages 109–160, 1964.
[4] B. Farb and D. Margalit. *A primer on mapping class groups*, volume 49. Princeton University Press, 2012.
[5] P. Hartman. On homotopic harmonic maps. *Canadian Journal of Mathematics*, 19(4):673, 1967.
[6] J. Hubbard and H. Masur. Quadratic differentials and foliations. *Acta Mathematica*, 142(1):221–274, 1979.
[7] N. Korevaar and R. Schoen. Global existence theorems for harmonic maps to non-locally compact spaces. *Comm. Anal. Geom*, 5:333–388, 1997.
[8] J. W. Morgan and P. B. Shalen. Valuations, trees, and degenerations of hyperbolic structures, i. *Annals of Mathematics*, pages 401–476, 1984.
[9] R. Schoen and S.-T. Yau. *Lectures on harmonic maps.* International Press Cambridge, 1997.
[10] M. Wolf. The teichmüller theory of harmonic maps. *Journal of differential geometry*, 29(2):449–479, 1989.
[11] M. Wolf. On realizing measured foliations via quadratic differentials of harmonic maps tor-trees. *Journal DAnalyse Mathematique*, 68(1):107–120, 1996.