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In this note, we will discuss some motivating examples to guide us to seek holomorphic objects when dealing with harmonic maps. This will lead us to a brief overview of the twistorial method for construction of harmonic maps from surfaces. We will use the sources [Woo87], [Hit82], [Sma92], and [BR90] as references.

Contents

1. Harmonic maps

1.1. Definitions and examples.

Let us recall the definition of a harmonic map and discuss a few simple examples, for completeness and motivation.

Definition 1.1. A smooth map $\phi : (M, g) \to (N, h)$ between Riemannian manifolds is *harmonic* if it is a critical point of the energy functional $E(\phi)$ given by

$$E(\phi) = \frac{1}{2} \int_M tr_g \phi^* h dvol_M .$$

The Euler-Lagrange equation for the energy functional is called the *harmonic map equation*:

$$\tau_\phi = tr_g(\nabla d\phi) = 0.$$

We call τ_ϕ the *tension field* of the map ϕ.

Examples of harmonic maps include:

(1) A constant speed parameterization $\gamma : \mathbb{R} \to (N, h)$ of a geodesic in N.
(2) A harmonic function $f : (M, g) \to \mathbb{R}$
(3) A holomorphic map $\phi : (M, J_M) \to (N, J_N)$ between complex manifolds.
(4) A parameterized surface $X : (\Sigma^2, g) \to \mathbb{R}^3$ is minimal if and only if X is harmonic and conformal. (This example will be revisited soon.)

Under pretty mild conditions, it is usually possible to find harmonic maps and establish their uniqueness (in a homotopy class). In this regard, when we have both the existence and uniqueness of a harmonic map, we can consider them as a natural candidate for a map between spaces. For example, for negatively curved target manifolds, a heat flow method can be applied to find harmonic maps. The model case is the theory of harmonic maps between two compact hyperbolic surfaces. In this case, there is a one-to-one correspondence between harmonic maps and homomorphisms of their fundamental groups (as the surfaces are $K(\pi, 1)$ spaces).

1.2. **Relating harmonic maps to holomorphic data.** When M is two dimensional, we can formulate an equivalent definition which associates harmonic maps to holomorphic data. To do this, let us introduce a local complex coordinate $z = x + iy$ on M. Then, up to a conformal factor, the tension field is equal to

$$\tau_\phi = \left(\phi^{-1}\nabla^N\right)_{\frac{\partial}{\partial \bar{z}}} \phi_* \left(\frac{\partial}{\partial z}\right)$$

Here, we have complexified the pull-back tangent bundle. In other words, we are considering the complexification of the total derivative, $(D\phi)^{\mathbb{C}}$, as a section of $TM \otimes (\phi^{-1}TN)^{\mathbb{C}}$. The harmonic map equation then reads

$$\left(\phi^{-1}\nabla^N\right)_{\frac{\partial}{\partial \bar{z}}} \phi_* \left(\frac{\partial}{\partial z}\right) = 0$$

or equivalently, the complex conjugate of the same equation

$$\left(\phi^{-1}\nabla^N\right)_{\frac{\partial}{\partial z}} \phi_* \left(\frac{\partial}{\partial \bar{z}}\right) = 0.$$

Before we go further, let us recall a classical theorem of Chern:

Theorem 1.1. *Let $\phi : M^2 \to \mathbb{R}^n$ be a conformal mapping. Then its Guass map $\gamma : M^2 \to Gr_2^{or}(\mathbb{R}^n)$ is antiholomorphic if and only if ϕ is harmonic.*

In light of this theorem, we are going to try to detect harmonic maps through associated holomorphic maps, which are more rigid and can be shown to exist by algebraic methods. At this point, let us record a few notes and observations in order to make sense of the statement of Chern's theorem:

(1) The Grassmannian space $Gr_2^{or}(\mathbb{R}^n)$ of oriented 2-planes in $\mathbb{R}^n$ has a natural complex structure induced by its immersion

$$\sigma : Gr_2^{or}(\mathbb{R}^n) \hookrightarrow \mathbb{C}P^n$$

into a space with a complex structure $J_{\mathbb{C}P^n}$. Here, where $X, Y \in T_0\mathbb{R}^n$ define the 2-plane $X \wedge Y$, we have $\sigma(X \wedge Y) := X + iY$.

(2) Note that $J := \sigma^*(J_{\mathbb{C}P^n})$ serves as an almost complex structure for $Gr_2^{or}(\mathbb{R}^n)$, and can be expressed locally as a rotation. At a point $P \in Gr_2^{or}(\mathbb{R}^n)$, let A_P denote the rotation through P by an angle of $\frac{\pi}{2}$, so that $A_P \circ A_P = -I$. There is an identification of the tangent spaces:

$$T_P Gr_2^{or}(\mathbb{R}^n) = Hom(P, P^\perp).$$

Using this identification, given a tangent vector $V \in Hom(P, P^\perp)$, we have $J(V) = V \circ A_P$. We easily see that $J^2(V) = -V$.

(3) The Gauss map is defined as one would expect: for every point $p \in M$, $\gamma(p)$ is the oriented 2-plane $D\phi(T_pM) \in Gr_2^{or}(\mathbb{R}^n)$, oriented to respect the orientation induced by $D\phi$.

(4) We call a map $\gamma : (M, J_M) \to (N, J_N)$ between almost-complex manifolds holomorphic when $D\phi$ intertwines the almost complex structures:

$$J_N \circ D\gamma = D\gamma \circ J_M$$

(5) A key observation is the identity:

$$\sigma(D\phi(\partial_x) \wedge D\phi(\partial_y)) = D\phi(\partial_x) + iD\phi(\partial_y)$$

Furthermore, $D\phi(\partial_x) + iD\phi(\partial_y) = D\phi(\partial_{\bar{z}}) \in \mathbb{C}P^n$, since $\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left[\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right]$.

With these observations in hand, let us sketch a proof of Chern's theorem.

Proof. The re-expression of the tension field of ϕ reveals that ϕ is harmonic precisely when $\partial_z(\frac{\partial \phi}{\partial \bar{z}}) = 0$,. Since $\sigma \circ \gamma(z) = \frac{\partial \phi}{\partial \bar{z}}(z)$, anti-holormophicity of the Gauss map is equivalent to $\partial_z(\frac{\partial \phi}{\partial \bar{z}}) = \lambda(z)\frac{\partial \phi}{\partial \bar{z}}$ for some smooth function $\lambda(z)$. Observe that $\partial_z(\frac{\partial \phi}{\partial \bar{z}})$ is a multiple of the mean curvature vector, so that it is perpendicular to both $\frac{\partial \phi}{\partial \bar{z}}$ and $\frac{\partial \phi}{\partial z}$. So, $\lambda(z) \equiv 0$. $\square$

We should appreciate this theorem, at least at face value, because it allows us to pass from finding harmonic maps to finding holomorphic curves (which may be easier to classify, etc.). Twistor theory attempts to generalize this theorem to general Riemannian manifold targets. We will soon discuss the case when the target N is a 4-dimensional Riemannian manifold, replacing the space of oriented 2-planes in $\mathbb{R}^n$ with $Gr_2^{or}(TN)$, the fiber bundle over N whose fiber at $y \in N$ is $Gr_2^{or}(T_yN)$. Let us first make put the situation into a general framework.

2. Twistor fibrations

Definition 2.1. A *twistor fibration* $\pi : Z \to N$ is a fibration of an almost complex manifold (Z, J^Z) over a Riemannian manifold N for which, for any holomorphic map $\psi : M \to Z$ from an almost Hermitian manifold with co-closed Kahler form, the composition $\pi \circ \psi : M \to N$ is harmonic. When these objects exist, we call ψ the *twistor lift* and ϕ the *twistor projection.*

The central problems pertaining harmonic maps in twistor theory are:

(1) Construct twistor fibrations $\pi : Z \to N$.
(2) Do there always exist twistor lifts of harmonic maps? More precisely, if given $\phi : M \to N$ harmonic, can we find $\psi : M \to (Z, J^Z)$ holomorphic such that $\phi = \pi \circ \psi$?
(3) When does the association of a harmonic map ϕ to its twistor lift ψ provide a 1-1 correspondence?
(4) Find holomorphic maps $M \to Z$. In our case, M will be a Riemann surface, so this amounts to finding holomorphic curves in Z. This actually lets us produce harmonic maps.

2.1. Motivation: Twistorial interpretation of the Weierstrass representation for minimal surfaces.

Harmonic maps play a prevalent role in the theory of minimal surfaces in $\mathbb{R}^3$ because Chern's theorem tells us that a parameterization of a surface is minimal if and only if it is conformal and harmonic. The classical development of studying minimal surfaces stagnated until the advent of the Weierstrass representation for minimal surfaces in $\mathbb{R}^3$. We will shortly describe the appearance of harmonic maps in this context and the holomorphic data associated to them.

The Weierstrass representation is given by a choice of holomorphic and meromorphic forms F and G on a Riemann surface Σ and then forming the parameterized surface

$$\vec{\phi}(z) = (\phi^1, \phi^2, \phi^3)(z) = \mathcal{R}\left(\int_{z_0}^{z} \frac{1}{2}\left(1 - G^2\right) F d\zeta, \int_{z_0}^{z} \frac{i}{2}\left(1 + G^2\right) F d\zeta, \int_{z_0}^{z} GF d\zeta\right).$$

The symbol $\mathcal{R}$ denotes taking the real part of the function, so $\vec{\phi}$ parameterizes a surface in $\mathbb{R}^3$. Note that in order to define this, we need to pick an arbitrary point $z_0 \in \Sigma$ to begin integrating from. It turns out that this surface is minimal! To see why, we will show that $\vec{\phi}$ is both conformal and harmonic.

Well, it turns out that $\vec{\phi}$ is conformal if and only if its Hopf differential (the (2,0)-part of the pullback metric) vanishes. We can apply the Fundamental Theorem of Calculus to compute this:

$$\left[\left(D\vec{\phi}\right)^* (dx^2 + dy^2 + dz^2)\right]^{(2,0)} = \left[\frac{1}{2}\left(1 - G^2\right) F\right]^2 + \left[\frac{i}{2}\left(1 + G^2\right) F\right]^2 + [GF]^2 = 0$$

That the map is harmonic is a simple consequence of the fact that each component function is the real part of a holomorphic function! (We'll assume the integrands have well-behaved poles.)

This parameterization originally came about by trying to exploit the fact that a harmonic function on $\mathbb{C}$ can be expressed uniquely as the real part of a holomorphic function, and has been useful in characterizing properties of complete minimal surfaces of finite total Guass curvature [Oss86]. It is worth pointing out that the Gauss map is expressible in terms of the holomorphic data as $\gamma = \chi \circ G$, where $\chi : S^2 \to \mathbb{R}^2$ denotes stereographic projection.

To relate this representation to a twistor projection, we need to modify the representation somewhat, working instead with its "Weierstrass representation in free form." This parameterizes the surface by its Gauss map variable G (see [Sma92] for more details). Let's begin by defining a new function f implicitly by

$$f'''(\zeta) = F \circ G^{-1}(\zeta) \frac{dG^{-1}}{d\zeta}(\zeta).$$

Then the Weierstrass representation in free form is given by taking the real part of

$$\begin{aligned}
\phi^1 \circ G^{-1}(z) &= \frac{1}{2}(1-z^2)f''(z) + zf'(z) - f(z) \\
\phi^2 \circ G^{-1}(z) &= \frac{1}{2}(1+z^2)f''(z) + izf'(z) + if(z) \\
\phi^3 \circ G^{-1}(z) &= zf''(z) - f'(z)
\end{aligned}$$

In free form, the metric and Guass curvature of the surface can be given in terms of just f''':

$$\begin{aligned}
ds^2 &= \frac{1}{2}|f'''(z)|^2(1+|z|)^2\mathcal{R}(dz \otimes d\overline{z}) \\
K(z) &= -\frac{8}{|f'''(z)|^2(1+|z|^2)^2}
\end{aligned}$$

For example, Enneper's surface has $F(z) = 1$ and $G(z) = z$ in the Weierestrass representation. In free form, Enneper's surface is determined by the data $f(\zeta) = \frac{1}{6}\zeta^3$. Geometrically speaking, to see this, we can invoke the uniqueness of minimal surfaces in $\mathbb{R}^3$ of -4π total Gaussian curvature with an order 3 end. This forces Enneper's surface to have these holomorphic data.

Let us now understand this from the twistorial point of view, as first recorded in [Hit82]. In order to do this, we will construct a twistor fibration Z over the Euclidean space $\mathbb{R}^3$ and then exhibit the correspondence by twistor lifts and projections between holorphic curves in Z with minimal surfaces in $\mathbb{R}^3$ given by the Weierstrass representation in free form.

Let Z be the space of oriented geodesics in $\mathbb{R}^3$. Two vectors $\mathbf{u} \in S^2$ and $\mathbf{v} \in \mathbb{R}^3$ are required to uniquely specify the geodesic line $\mathbf{v} + t \cdot \mathbf{u} \subset \mathbb{R}^3$, thinking of $\mathbf{v}$ as the point on the geodesic closest to the origin and of $\mathbf{u}$ as the direction of the geodesic. So, we have a parameterization of $Z \cong \{(u, v) \in S^2 \times \mathbb{R}^3 | \mathbf{u} \cdot \mathbf{v} = 0\}$, where we think of S^2 as the unit sphere in $\mathbb{R}^3$. It turns out that Z is diffeomorphic to the tangent bundle of the projectivized complex line, $T\mathbb{C}P^1$, naturally equipped with its holomorphic structure. The induced complex structure J can be realized by acting as the differential of the map on Z which reverses the orientation on each line.

Now, suppose we have a minimal surface given by $\vec{\phi}$. Then there exists a unique holomorphic map $\vec{\mathbf{x}} : \Sigma \to \mathbb{C}^3$ so that

$$\phi(z) = \vec{\mathbf{x}}(z) + \overline{\vec{\mathbf{x}}(z)}.$$

Well, $\vec{\mathbf{x}}$ is a holomorphic curve η in $\mathbb{C}^3$ because of holomorphicity of ϕ, and it is even a null curve because of conformality of ϕ (the Hopf differential of ϕ is precisely $\sum |\frac{\partial \vec{\mathbf{x}}}{\partial z}|$).

On the other hand, suppose we have a holomorphic curve in Z. Note that the point $x \in \mathbb{R}^3$ corresponds to a holomorphic section of $\pi : Z \to \mathbb{C}P^1 = S^2$, given by all the lines (determined by their direction) that go through x. This section will be of degree 2 because the tangent bundle of $\mathbb{C}p^1$ has degree 2. Then we can locally compute the osculating section to it, which will have the form

$$f(\omega) = a + b\omega + c\omega^2,$$

for some parameterizing neighborhood $\omega \in U\mathbb{C}$ of the osculated point. We have:

$$\begin{aligned}
a + b\omega + c\omega^2 &= f(\omega) \\
b + 2c\omega &= f'(\omega) \\
2c &= f''(\omega)
\end{aligned}$$

Well, this data describes a holomorphic curve in $\mathbb{C}^3$ because f is holomorphic:

$$(a, b, c) = (f - \omega f' + \frac{1}{2}\omega^2 f'', f' - \omega f'', \frac{1}{2}f'').$$

Furthermore it will be a null curve because

$$(b')^2 - 4a'c' = \omega(f''')^2 - 2f'''(\frac{1}{2}\omega^2 f''') = 0.$$

Following [Hit82], we finish this computation by recognizing the identification of $\mathbb{R}^3$ inside Z. Any section $s(\omega)$ of Z can be expressed as

$$s(\omega) = ((x+iy) - 2z\omega - (x-iy))\frac{\partial}{\partial\omega}.$$

Finally, re-expressing this in standard coordinates recovers the Weierstrass representation in free form.

3. Example: Generalizing Chern's Theorem

On a four dimensional Riemannian manifold N^4, let us consider the fiber bundle of oriented 2-planes $\pi : Gr_2^{or}(TN) \to N$ over it: over each point $y \in N$ is the fiber $Gr_2^{or}(T_yN)$. Given a conformal map $\phi : M^2 \to N$, we can define its *Gauss lift* as the map $\psi : M \to Gr_2^{or}(TN)$ by taking $\psi : x \mapsto D\phi|_x(T_xM)$. Now, in order to understand the relationship between harmonicity of ϕ and holomorphicity of ψ, have need to find an almost complex structure on $Gr_2^{or}(TN)$

The Levi-Civita connection ∇^N of N and the differential of the projection $D\pi$ allow us to decompose the tangent bundle $TGr_2^{or}(TN)$ into horizontal and vertical sub-bundles: pointwise, this amounts to

$$T_K Gr_2^{or}(TN) = im(D\pi|_{\pi(K)}) \oplus ker(D\pi|_{\pi(K)}),$$

and we will write $TGr_2^{or}(TN) = \mathcal{H} \oplus \mathcal{V}$. Now, in order to give $Gr_2^{or}(TN)$ an almost complex structure, it suffices to find an almost complex structure on each of the sub-bundles $\mathcal{H}$ and $\mathcal{V}$.

Each fiber of the vertical space $\mathcal{V}$ is a vector space on which $O(2n)$ acts by conjugation: for $g \in O(2n)$, we have the action $g \cdot J = gJg^{-1}$. The stabilizer of this action is $U(n)$, so we have a natural isomorphism $J_x(N) \cong O(2n)/U(n)$. This isomorphism endows $\mathcal{V}$ with an almost complex structure J^V coming from the fact that $O(2n)/U(n)$ is a Hermitian symmetric space.

Proposition 3.1. *Let $\phi : M^2 \to N^4$ be a conformal map. Then ϕ is harmonic if and only if its Gauss lift $\psi : M^2 \to Gr_2^{or}(TN)$ is vertically anti-holomorphic, i.e., that $J^V \circ D\psi = (D\psi \circ J^M)^V$ where the superscript ^{V} denotes projection onto $\mathcal{V}$.*

On the horizontal spaces, we can use the natural almost complex structure on $Gr_2^{or}(TN)$ and push it forward to N. To see this, take a point $W \in Gr_2^{or}(TN)$. Let J_W^H denote the rotation by $\frac{\pi}{2}$ on W and $W^\perp$, considered as subspaces of $T_{\pi(W)}(N)$. Note that this is compatible with orientation, and we can denote the globally defined almost complex structure on $\mathcal{H}$ by J^H. Morally, ϕ is conformal if and only if its Gauss lift is horizontally anti-holomorphic, as the choice of horizontal almost complex rendered it so.

Proposition 3.2. *Let $\phi : M^2 \to N^4$ be a smooth immersion. Then ϕ is conformal if and only if its Gauss lift is horizontally holomorphic, i.e., that $J^H \circ D\psi = (D\psi \circ J^M)^H$ where the superscript ^{V} denotes projection onto $\mathcal{H}$.*

With these almost complex structures on sub-bundles, we can form two almost complex structures, $J_1 := J^H + J^V$ and $J_2 := J^H - J^V$. (Note the reversal of the sign on J^V to circumvent stating anti-holomorphicity versus holomorphicity in certain theorems.) We can finally state:

Theorem 3.3. *A smooth immersion $\phi : M^2 \to N^4$ is conformal and harmonic if and only if its Gauss lift $\psi : M^2 \to Gr_2^{or}(TN)$ is J_2-holomorphic.*

4. Example: Almost complex structures on an even-dimensional manifold

Consider an even-dimensional Riemannian manifold N^{2n}. Let $\pi : J(N) \to N$ be the space of almost complex structures compatible with the metric on N. The fibers are given by

$$J_x(N) = \{J \in End(T_xN) | J^2 = -I, J \text{ skew-symmetric}\}.$$

We can define analogous almost complex structures on $J(N)$ as above.

Theorem 4.1. *A map $\phi : M \to N^{2n}$ of a Riemann surface has a J_2-holomorphic lift $\psi : M \to J(N)$ if and only if it is weakly conformal, harmonic, and $\phi^*((w_1(N))) = 0$.*

It is worth noting that the almost complext manifolds $(J(N), J_1)$ and $(J(N), J_2)$ are not always well-behaved. J_2 is never integrable. On the other hand, we have the following concerning J_1:

Proposition 4.2. *Let $j \in J(N)$ with $\sqrt{-1}$-eigenspace $T^+ \subset T_{\pi(j)}N^{\mathbb{C}}$. Let R denote the Riemann curvature tensor of ∇^N. Then the Nijenhuis tensor of J_1 vanishes at j if and only if $R(T^+, T^+)T^+ \subset T^+$*

It follows from this proposition that J_1 is integrable if and only if N is conformally flat. Nonetheless, this leads us to find better twistor spaces which reflect the geometry of N.

5. Twistor fibrations over symmetric spaces

It turns out that there are reasonably nice twistor spaces for N a symmetric space.

Proposition 5.1. *Suppose G/K is an inner symmetric space, i.e., rank(G) = rank(K). If $H \subset K$ is a centralizer of a torus, then the homogeneous fibration $\pi : (G/H, J^2) \to G/K$ is a twistor fibration.*

Proof. There is at least one parabolic subgroup P of the complexified Lie group $G^{\mathbb{C}}$ such that $P \cap G = H$, so that $G/H \cong G^{\mathbb{C}}/P$. This diffeomorphism induces a complex structure J^1 on G/H coming from $G^{\mathbb{C}}/P$. Similar to before, reverse the orientation of the complex structure on the vertical spaces. □

Theorem 5.2. *Let $\phi : S^2 \to G/K$ be a harmonic map. Then there is a centralizer $H \subset K$ of a torus in G such that ϕ has a J^2-holomorphic lift $S^2 \to G/H$ into the twistor space $(G/H, J^2)$.*

Definition 5.1. *Complex flag manifolds* are the homogeneous spaces G/H, where H is the centralizer of a torus.

6. Getting new harmonic maps from a given one

6.1. **Flag transform.** Any harmonic map of the 2-sphere into a compact Lie group of type H can be obtained by repeated flag transforms of a constant map (which is harmonic). (See Theorem 8.9 in [BR90] for details and definitions.)

6.2. **Uhlenbeck's extended solutions via loop groups.** There is a left-invariant almost complex structure on ΩG which coincides with the Kahler structure on the pseudo-horizontal distribution and makes the map $\Omega G \to G$ given by evaluation at -1 into a twistor fibration. (See Proposition 8.16 in [BR90] for more information.)

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