Survey of Taubes' "$PSL(2, \mathbb{C})$ connections on 3-manifolds with L^2 bounds on curvature"

Laura Fredrickson

January 8, 2015

Taubes' paper "$PSL(2, \mathbb{C})$ connections on 3-manifolds" is the first paper in a series[1] of (currently 3) papers that generalize Uhlenbeck's compactness theorem when the gauge group G is not compact. In this paper, Taubes generalizes data and conditions, which are natural to the special case of solutions of Hitchin's equations on Riemann surfaces, to the more general 3-dimensional settting. These results about 3-manifolds and techniques used are a stepping stone to results about higher-dimensional manifolds. For example, understanding the solutions of the Kapustin-Witten equations on $M^3 \times \mathbb{R}$ is a motivating application.

This talk is a survey of Taubes paper, focusing on the aspects that are likely more interesting to people at this workshop. All errors are my own. Roughly, the talk is outlined as follows:

- **Taubes' theorem as a generalization of Uhlenbeck Compactness:** After stating Uhlenbeck's compactness theorem, we will compare Uhlenbeck's and Taubes' respective theorems. There are two very different cases in Taubes theorem, depending on whether some "energy," $\int_M |\phi_n|^2$, is uniformly bounded or not. If the "energy" is bounded, Taubes' theorem is a straightforward application of Uhlenbeck's theorem. However, if the "energy" is unbounded, Taubes' theorem is a new and interesting result.

[1]These three papers are:
(1) "$PSL(2, \mathbb{C})$ connections on 3-manifolds" May 2012 (substantial revision July 2014), arXiv:1205.0514
(2) "Compactness theorems for $SL(2, \mathbb{C})$ generalizations of the 4-dimensional anti-self-dual equations" 2013 (substantial revision July 2014) arXiv: 1307.6447
(3) "The zero loci of $\mathbb{Z}/2$ harmonic spinors in dimensions 2,3,4" July 2014, arXiv: 1407.6206

- **Interpretation of Taubes' theorem for 2-manifolds:** The statement of Taubes' theorem when the "energy" is unbounded is complicated for 3-manifolds. However, when the base manifold is a Riemann surface, the data and the conclusions are easier to understand because they are related to a certain holomorphic differential.

- **Data for 3-manifolds & Almgren's frequency function:** We introduce Almgren's frequency function, an important analytic tool that serves as an (inferior) replacement of the holomorphic quadratic differential.

- **Comments on Taubes' Proof:** Finally, in this short section, I'll say something about Taubes' proof and why the limiting configuration has some of the properties that it does.

1 Taubes' theorem as a generalization of Uhlenbeck Compactness

1.1 Uhlenbeck Compactness

Because Taubes' theorem is a generalization of Uhlenbeck's theorem, we turn our attention first to Uhlenbeck's original theorem in her paper "Connections with L^p bounds on curvature" which has been so foundational for gauge theory:

Theorem 1.1 *(Weak Uhlenbeck Compactness) Let M be a compact Riemannian n-manifold, let G be a compact Lie group, and let $P \to M$ be a principal G-bundle. Let $1 < p < \infty$ be such that $p > \frac{n}{2}$. Let $\{A_n\}$ be a sequence of $L^{p,1}$ connections on P. Suppose $\{\int_M |F_{A_n}|^p\}$ is bounded. Then there is a subsequence of $\{A_n\}$, (again denoted $\{A_n\}$), and a corresponding sequence of $W^{2,p}$ gauge transformations $\{g_n\}$ such that*

$$\{g_n^* A_n\} \rightharpoonup^{W^{1,p}} A_\diamond,$$

and $A_\diamond$ is an $W^{1,p}$ connection on P.

1.2 Set-up for Taubes' Theorem

Taubes' Theorem has a similar set-up as Uhlenbeck's Theorem with $p = 2$.
Set-up:

- M^3 a compact Riemannian 3-manifold. (Or, Σ^2 a compact Riemann surface.)
- Let $G = PSU(2)$ (hence $G_{\mathbb{C}} = PSL(2,\mathbb{C})$).
- Let $P \to M$ (or Σ) be a principal G-bundle.
- Let $\mathbb{A}_n = A_n + i\phi_n$ be a sequence of complex connections on $P \times_G G_{\mathbb{C}}$.

As we pass from real connections to complex connections, we'll need a new notion of a bound on curvature, and more.

1.2.1 A hermitian structure

Note that our $G_{\mathbb{C}}$ bundle, $P \times_G G_{\mathbb{C}}$, has a G-subbundle P. This reduction of structure group from $G_{\mathbb{C}}$ to G gives a hermitian metric on the associated complex vector bundle. We get a splitting of $\mathfrak{g}_{\mathbb{C}}$ into unitary ($\mathfrak{g}$) and hermitian (here, $i\mathfrak{g}$) components. Consequently, we get a splitting of the complex connection

$$\mathbb{A} = A + i\phi$$

with $A, \phi \in \Omega^1(M, \mathfrak{g}_P)$.

1.2.2 An aside: Hitchin's equations

Hitchin's equation on a Riemann surface motivate the symbols and definitions in this talk. Here we briefly review Hitchin's equations to fix notion. Let P be a principle $PSU(2)$-bundle on Σ^2. A pair $(A, \phi) \in \Omega^1(\Sigma, \mathfrak{g}_P) \times \Omega^1(\Sigma, \mathfrak{g}_P)$ is a solution of Hitchin's equations if:

$$\begin{aligned} F_A - \phi \wedge \phi &= 0 \\ d_A \phi &= 0 \\ d_A^* \phi &= 0. \end{aligned}$$

The first two equations defining the Hitchin moduli space can be repackaged as the real and imaginary parts of the complex curvature $\mathcal{F}_{A+i\phi} = 0$. The third equation is often called the "zero moment map condition."

This unitary formulation of Hitchin's equations that Taubes uses is somewhat less common than the more standard holomorphic Higgs bundle formulation of Hitchin's equations:

$$\begin{aligned} F_A - [\varphi, \varphi^\dagger] &= 0 \\ \overline{\partial}_A \varphi &= 0 \\ \partial_A \varphi^\dagger &= 0. \end{aligned}$$

However, the unitary language is better-suited for the isomorphism between moduli space of solutions of Hitchin's equations (up to real gauge $G = PSU(2)$ transformations) and the moduli space of flat $G_{\mathbb{C}} = PSL(2, \mathbb{C})$-connections (up to complex gauge transformations). Moreover, the holomorphic formulation certainly doesn't make sense on the 3-manifolds, as in Taubes' paper.

Remark Note that this real Higgs field ϕ is related to the usual Higgs field φ by $\varphi = \phi^{(1,0)}$. Consequently, the holomorphic quadratic differential is $\varphi_2 = \det \varphi = \det \phi^{(1,0)}$. The flat connection is $\mathbb{A} = A + \varphi + \varphi^\dagger$.

Note: Taubes writes a complex connection as $\mathbb{A} = A + i\mathfrak{a}$, splitting the connection into real and imaginary parts. However, because I don't want to obscure the relationship to the more familiar Hitchin's equations, I'll to use ϕ rather than Taubes' $\mathfrak{a}$.

1.2.3 A notion of "energy"

The correspondence between flat $PSL(2, \mathbb{C})$ connections (up to complex gauge transformations) on a Riemann surfaces and solutions of Hitchin's equations (up to real gauge transformations) is mediated through the existence of a so-called harmonic metric, a minimizer of the energy

$$E(A, \phi) = \int_M |\phi|^2.$$

This same energy will play an important role in Taubes' theorem. We briefly discuss the harmonic metric because the Euler-Lagrange equations ($d_A^* \phi = 0$ and $d_A \phi = 0$) for the energy $\int_M |\phi|^2$ appeared in Hitchin's equations and will appear again and again.

Let $P_{\mathbb{C}}$ be a principal $SL(2, \mathbb{C})$ bundle on Σ, forgetting the reduction of structure group in the previous section for the moment. A flat connection $\mathbb{A}$ on $P_{\mathbb{C}}$ (up to complex gauge transformations) is equivalent to a representation $\rho : \pi_1(\Sigma) \to SL(2, \mathbb{C})$ (up to conjugation). A hermitian metric on the associated vector bundle $P_{\mathbb{C}} \times_{SL(2,\mathbb{C})} \mathbb{C}^2$ is given by a ρ-equivariant map $h : \tilde{\Sigma} \to SL(2,\mathbb{C})/SU(2) \cong \mathbb{H}^3$. This "energy" of this map is

$$\mathcal{E}_\rho[h] = \int_\Sigma |Dh|^2.$$

The hermitian metric also gives a splitting of the complex flat connection $\mathbb{A}$ into unitary and skew-unitary components, $\mathbb{A} = D_A + i\phi$. The "energy" of the hermitian metric can also be written in terms of the h-skew-unitary part of the connection:

$$\mathcal{E}_{(\rho)}[h] := 4 \int_\Sigma |\phi|^2.$$

Note that this latter definition generalizes when $\mathbb{A}$ is not flat, and this is the definition of "energy" Taubes takes.

The Euler-Lagrange equations of the energy functional are

$$\begin{aligned} d_A^* \phi &= 0 \\ d_A \phi &= 0. \end{aligned}$$

Though the connections we consider are not necessarily flat the "energy" $|\phi|^2_{L^2(M)}$, the first order $(d_A^* + d_A)\phi$ operator from the Euler-Lagrange equation, and the associated second order Laplacian $(d_A d_A^* + d_A d_A^*)\phi$ will play an important role in Taubes' paper.

1.2.4 A generalized bound on "curvature"

Given a complex connection $\mathbb{A}$ on $P \times_G G_{\mathbb{C}}$, the analog of Uhlenbeck's bound on $\{\int_M |F_{A_n}|^2\}$, is *not* a bound on the complex curvature $\int_M |\mathcal{F}_{\mathbb{A}}|^2$, but rather a bound on the functional

$$\mathfrak{F}(\mathbb{A}) = \mathfrak{F}(A + i\phi) = \int_M \left(|F_A - \phi \wedge \phi|^2 + |d_A \phi|^2 + |d_A^* \phi|^2 \right).$$

(We are using the hermitian metric from the implicit reduction of structure group $P \times_G G_{\mathbb{C}}$ to decompose $\mathbb{A} = A + i\phi$ is into unitary and skew-unitary parts.) Note that the functional features the term $d_A^* \phi$ discussed in the previous section.

1.3 Case I: Bounded energy

Taubes considers a sequence of complex connections $\mathbb{A}_n = A_n + i\phi_n$ on $P \times_{PSU(2)} PSL(2,\mathbb{C})$. There are two very different cases, depending on whether $||\phi||_{L^2(M)}$ has a bounded subsequence or not.

Theorem 1.2 *([T1] Thm 1.1 Case I) Let P be a principal $PSU(2)$ bundle on M^3 (or Σ^2). Let $\mathbb{A}_n = A_n + i\phi_n$ be a sequence of connections on $P \times_{PSU(2)} PSL(2,\mathbb{C})$ such that the corresponding sequence $\{\mathfrak{F}(A_n + i\phi_n)\}$ is bounded. If $||\phi_n||_{L^2(M)}$ has a bounded subsequence, then there is a subsequence $\mathbb{A}_n$, hence renumbered consecutively from 1, and a corresponding sequence of real gauge transformations $g \in Aut(P)$ such that*

$$g_n^* \mathbb{A}_n \rightharpoonup^{W^{1,2}} \mathbb{A}_\diamond.$$

This limit, $\mathbb{A}_\diamond$ is a $W^{1,2}$ connection on $P \times_{SO(3)} PSL(2,C)$ with

$$\mathfrak{F}(A_\diamond + i\phi_\diamond) \leq \limsup \mathfrak{F}(A_n + i\phi_n).$$

What does this theorem say for a sequence of reductive flat connections on a Riemann surface? Suppose we are given a sequence $\{\mathbb{A}_n\}$ of flat reductive $PSL(2,\mathbb{C})$ connections on Σ^2. Fixing a metric h on the bundle, we get a decomposition $\mathbb{A}_n = A_n + i\phi_n$. Depending on the choice of metric (i.e. the reduction of structure group), the sequence $\mathfrak{F}(A_n + i\phi_n) = \int_\Sigma |d_A^*\phi|^2$ need not be bounded. However, if it is bounded, Taubes' theorem gives a subsequence and corresponding sequence of real gauge transformations such that $g^*(A_n + i\phi_n) \rightharpoonup A_\diamond + i\phi_\diamond$. This limiting connection $A_\diamond + i\phi_\diamond$ is certainly flat. However, as h was not the harmonic metric for the connections $\mathbb{A}_n$, it is not necessarily the harmonic metric for the limiting connection $\mathbb{A}_\diamond = A_\diamond + i\phi_\diamond$.

What tools are required to prove the theorem? We will need the following two tools

- Uhlenbeck Compactness, and
- an integral version of the Bochner-Weitzenböck formula.

1.3.1 Bochner-Weitzenböck Formula

Given a connection D_A, a Bochner-Weitzenböck formula relates the extension of D_A to d_A, acting on p-forms,

$$d_A : \Omega^p(M, \mathfrak{g}_P) \to \Omega^{p+1}(M, \mathfrak{g}_P),$$

and the extension of D_A to ∇_A, acting on p-tensors,

$$\nabla_A : \Gamma((T^*M)^{\otimes p} \otimes \mathfrak{g}_P) \to \Gamma((T^*M)^{\otimes p+1} \otimes \mathfrak{g}_P).$$

Because 1-forms and 1-tensors are the same, the Laplacians

$$d_A^* d_A + d_A d_A^* : \Omega^1(M, \mathfrak{g}_P) \to \Omega^1(M, \mathfrak{g}_P)$$

and

$$\nabla_A^* \nabla_A : \Gamma(T^*M \otimes \mathfrak{g}_P) \to \Gamma(T^*M \otimes \mathfrak{g}_P)$$

act on the same space.

Given $\phi \in \Omega^1(M, \mathfrak{g}_P) \cong \Gamma(T^*M \otimes \mathfrak{g}_P)$, the **Bochner-Weitzenböck formula** expresses the relation between these two Laplacians:

$$(d_A^* d_A + d_A d_A^*)\,\psi = \nabla_A^* \nabla_A \psi + \star\,[\star F_A, \psi] + \mathrm{Ric}(\langle \cdot \otimes \psi \rangle). \tag{1}$$

The operator $\star : \Omega^k(M^n) \to \Omega^{n-k}(M^n)$ is the Hodge star operator, $\mathrm{Ric} \in \Gamma(\otimes^2 TM)$ is the Ricci tensor (with indices raised using the metric on M), and $\langle \cdot, \cdot \rangle$ is the unitary-gauge-invariant inner product.

Let f be any C^2 function on X and $R \in (0, \infty)$ be a positive parameter. The integral Bochner-Weitzenböck formula that Taubes uses is

$$\begin{aligned} &\int_X f \left(|d_A\psi|^2 + |d_A^*\psi|^2 + R^{-2} \left| F_A - \frac{R^2}{2}[\psi, \psi] \right|^2 \right) \qquad (2) \\ = &\int_x \frac{1}{2} d^* df |\psi|^2 + \int_x f \left(|\nabla_A \psi|^2 + R^2 |[\psi, \psi]|^2 + R^{-2}|F_A|^2 + \mathrm{Ric}(\langle \psi \otimes \psi \rangle) \right) \\ &- \int_X \langle df, \psi(d_A^*\psi) \rangle + df \wedge \psi \wedge \star d_A \psi. \end{aligned}$$

Note that this equation was obtained by taking the inner product of the Bochner-Weitzenböck formula with the 1-form/1-tensor $f\psi$ then integrating by parts. Consequently, the integral identities only capture information in the ψ direction, so there's quite a bit more information in the Bochner-Weitzenböck formula than the integral Bochner-Weitzenböck formula.

Remark In Taubes application of the integral Bochner-Weitzenböck formulas, $(\psi, R) = (\phi, 1)$ or–more often– $(\phi, R) = \left(\frac{\phi}{||\phi||_{L^2(M)}}, ||\phi||_{L^2(M)} \right)$. The function f is either 1 or some non-singular approximation of G_p, the fundamental solution of $-d^*d - 1 = \Delta - 1$, i.e. $(\Delta - 1)G_p = \delta_p$ for $p \in M$.

1.3.2 Sketch of Proof

The proof of Theorem 1.2 is almost a direct application of Uhlenbeck compactness.

From an integral form of the Bochner-Weitzenböck formula, one can obtain the following equation [T1, Eq. 2.4]

$$\int_M \left(|\nabla_A \phi|^2 + |F_A|^2\right) \leq \mathfrak{F}(A + i\phi) + c_0 \int_M |\phi|^2,$$

and consequently, bounds on $\mathfrak{F}(A+i\phi)$ and $||\phi||_{L^2(M)}$ give bounds on $||\nabla_A \phi||_{L^2(M)}$ and $||F_A||_{L^2(M)}$. Given the bounds on $||F_A||_{L^2(M)}$, Uhlenbeck compactness gives a subsequence and sequence of gauge transformations such that $g_n^* A_n$ converge weakly in $W^{1,2}$. Uhlenbeck compactness does not guarantee that $g_n^* \phi_n$ converges. However, the bound on $||\nabla_{A_n} \phi_n||_{L^2(M)}$ implies that there is a subsequence (of the subsequence) such that $g_n^* \phi_n$ converges weakly in $W^{1,2}$.

1.4 Case II: Unbounded energy

If there is no bounded subsequence, the situation is very different. It will be convenient to introduce a renormalized Higgs field

$$\hat{\phi} = \frac{\phi}{||\phi||_{L^2(M)}}.$$

Theorem 1.3 *([T1] Thm 1.2 Case II for* 3*-manifolds) Let* P *be a principal* $PSU(2)$*-bundle on* M^3*. Let* $\mathbb{A}_n = A_n + i\phi_n$ *be a sequence of connections on* $P \times_{PSU(2)} PSL(2, \mathbb{C})$ *such that the corresponding sequence* $\{\mathfrak{F}(A_n + i\phi_n)\}$ *is bounded. If* $||\phi_n||_{L^2(M)}$ *has* no *bounded subsequence,* <u>*then*</u> *there is a subsequence* $\mathbb{A}_n$*, hence renumbered consecutively from 1, a corresponding sequence of* real *gauge transformations* $g \in Aut(P)$*, and the following additional data:*

- *a closed set,* $Z \subset M$*,*
- *a real line bundle* $\mathcal{I} \to M - Z$*, and*
- *a harmonic section,* μ*, of* $\mathcal{I} \otimes T^*M \to M - Z$*, the space of* $\mathcal{I}$*-valued* 1*-forms.*

This data satisfies:

- $Z \subset \mu^{-1}(0)$*, which is contained in a countable union of* 1*-dimensional Lipschitz curves. (Note: One can actually say a bit more than this.)*
- $g_n^* A_n \rightharpoonup^{W^{1,2}_{loc}(M-Z)} A_\diamond$*, a* $W^{1,2}_{loc}$ *connection on* $P|_{M-Z}$
- $g_n^* \hat{\phi}_n \rightharpoonup^{W^{1,2}_{loc}(M-Z)} \hat{\phi}_\diamond$ *such that* $\hat{\phi}_\diamond^{(1,0)} = \mu\sigma$ *where* σ *is a unit-length,* $A_\diamond$*-covariantly constant section of* $(\mathcal{I} \otimes P) \times_{\mathbb{Z}_2 \times PSU(2)} \mathfrak{su}(2)$*.*

What is the meaning of this data and conditions? Fundamentally, the data and conditions above are the more obtuse 3-dimensional generalization of the more obvious data and conditions on Riemann surfaces. In the next section we discuss the corollary for Riemann surfaces before returning to the 3-dimensional case.

2 Interpretation of Taubes' Theorem for 2-manifolds

We state the corollary of Taubes theorem for 2-manifolds Σ. Note that the canonical isomorphism between $T\Sigma$ and $T^{(1,0)}\Sigma$ allows us to state the data and conditions in holomorphic–rather than harmonic–language.

Theorem 2.1 *([T1] Thm 1.2 Case II for 2-manifolds & Lemma 7.1) Let P be a principal $PSU(2)$-bundle on Σ, a compact, oriented Riemann surface. Let $\mathbb{A}_n = A_n + i\phi_n$ be a sequence of connections on $P \times_{PSU(2)} PSL(2,\mathbb{C})$ such that the corresponding sequence $\{\mathfrak{F}(A_n + i\phi_n)\}$ is bounded. If $||\phi_n||_{L^2(\Sigma)}$ has* no *bounded subsequence,* then *there is:*

- *a subsequence $\mathbb{A}_n$, hence renumbered consecutively from 1, and*
- *a corresponding sequence of* real *gauge transformations $g \in Aut(P)$,*

in addition to the following data:

- *a non-trivial holomorphic quadratic differential $\varphi_{2,\diamond}$*

which further gives the following data

- *Z, the zero set of $\varphi_{2,\diamond}$,*
- *$\tilde{\Sigma}$, the spectral cover of Σ (defined as the roots of $\varphi_{2,\diamond}$), which is an honest $\mathbb{Z}_2$-principle bundle on $\Sigma - Z$, and*
- *a real line bundle, $\mathcal{I}$, on $\Sigma - Z$, which is the associated bundle to the spectral cover, i.e. $\mathcal{I} = \tilde{\Sigma}|_{\Sigma - Z} \times_{\mathbb{Z}_2} \mathbb{R}$.*

This data satisfies:

- $g_n^* A_n \rightharpoonup^{W^{1,2}_{loc}(\Sigma - Z)} A_\diamond$*, a $W^{1,2}_{loc}$ connection on $P|_{\Sigma - Z}$*
- $g_n^* \frac{\phi_n}{||\phi_n||_{L^2(\Sigma)}} \rightharpoonup^{W^{1,2}_{loc}(\Sigma - Z)} \hat{\phi}_\diamond$.

This limiting configuration $(A_\diamond, \phi_\diamond)$ *satisfies*

$$\begin{aligned} \hat{\phi}_\diamond \wedge \hat{\phi}_\diamond &= 0 \\ d_{A_\diamond}\hat{\phi}_\diamond &= 0 \\ d^*_{A_\diamond}\hat{\phi}_\diamond &= 0. \end{aligned}$$

Consequently, $\hat{\phi}_\diamond^{(1,0)} = \sqrt{\phi_2}\sigma$ *where* σ *is a unit-length,* $A_\diamond$*-covariantly constant section of* $(\mathcal{I} \otimes P) \times_{\mathbb{Z}_2 \times PSU(2)} \mathfrak{su}(2)$*.*

Let's assume for the time being that the limiting configuration $(A_\diamond, \phi_\diamond)$, satisfies the three equation:

$$\begin{aligned} \hat{\phi}_\diamond \wedge \hat{\phi}_\diamond &= 0 \\ d_{A_\diamond}\hat{\phi}_\diamond &= 0 \\ d^*_{A_\diamond}\hat{\phi}_\diamond &= 0. \end{aligned}$$

Note that the holomorphicity of the associated quadratic differential, $\varphi_{2,\diamond} = \mathrm{tr}\left((\hat{\phi}^{(1,0)\,2} \right)$, follows from $d^*_{A_\diamond}\hat{\phi}_\diamond = 0$.

In the next subsection we'll say something about the spectral cover $\tilde{\Sigma}$, $\mathcal{I}$, and Z associated to $\varphi_{2,\diamond}$. We'll also say something about σ.

2.1 Explanation of $\mathcal{I}$

Given a holomorphic quadratic differential $\varphi_{2,\diamond}$, the spectral cover, $\tilde{\Sigma}$, of Σ is

$$\tilde{\Sigma}_{\varphi_{2,\diamond}} = \{\xi \in K_\Sigma : \xi^2 - \varphi_{2,\diamond} = 0.\}$$

It is ramified over Z, the zeros of $\varphi_{2,\diamond}$. However, on $\Sigma - Z$, the spectral cover is an honest $\mathbb{Z}_2$-principle bundle. If $\varphi_{2,\diamond}$ has a zero of even order at $p \in \Sigma$, then the punctured spectral cover is a trivial $\mathbb{Z}_2$ bundle near p. However, if $\varphi_{2,\diamond}$ has a zero of odd order at p, the punctured spectral cover is not trivial near p.

Define $\mathcal{I}$ to the be the real bundle associated to the spectral cover. The real bundle is trivial near zeros of $\varphi_{2,\diamond}$ of even order, and non-trivial near zeros of $\varphi_{2,\diamond}$ of odd order.

What's the use of $\mathcal{I}$*?* In general, a holomorphic quadratic differential is not the square of something in $T^*\Sigma$, hence the square root $\sqrt{\varphi_2}$ is only defined up to sign. In Taubes' notation, $\sqrt{\varphi_2}$ is a bona fide $\mathcal{I}$-valued 1-form, a section of $\mathcal{I} \otimes_{\mathbb{R}} K$. Alternatively, rather than viewing $\mathcal{I}$ as a bundle over Σ, we can view $\mathcal{I}$ as a bundle over Σ. Saying that the square root $\sqrt{\varphi_2}$ is a

$\mathcal{I}$-valued 1-form on Σ translates, in this alternative view, to the statement that $\sqrt{\varphi_2}$ (or more precisely, its pullback) is an honest 1-form on the spectral cover, $\tilde{\Sigma}$

2.2 A "preferred direction" of the Higgs field: σ

From $\hat{\phi}_\diamond \wedge \hat{\phi}_\diamond = 0$, we see that the Higgs field has a "preferred direction." What do I mean by this? In local coordinates $\{x, y\}$ on a patch of $\Sigma - Z$, the Higgs field is $\hat{\phi}_\diamond = \hat{\phi}_x dx + \hat{\phi}_y dy$ for $\hat{\phi}_x, \hat{\phi}_y \in \mathfrak{su}(2)$. The equation

$$0 = \hat{\phi}_\diamond \wedge \hat{\phi}_\diamond = [\hat{\phi}_x, \hat{\phi}_y] dx \wedge dy$$

guarantees that $\hat{\phi}_x$ and $\hat{\phi}_y$ commute. Hence, $\hat{\phi}_x$ and $\hat{\phi}_y$ are in the same $\mathfrak{u}(1) \subset \mathfrak{su}(2)$, which we call the "preferred direction." Let σ be a unit-length vector pointing in this "preferred direction" of the Higgs field. Note that because the intersection of the unit sphere in $\mathfrak{su}(2) \cong \mathbb{R}^3$ and $\mathfrak{u}(1) \cong \mathbb{R}$ consists two points (rather than one), σ is also only defined up to sign. The bundle $\mathcal{I}$ perfectly captures this sign ambiguity and so σ is valued in $(\mathcal{I} \otimes P) \times_{\mathbb{Z}_2 \times PSU(2)} \mathfrak{su}(2)$.

Any $\mathfrak{g}_P$-valued section, s, can be decomposed into directions parallel and perpendicular to σ, denoted $\pi_{<\sigma>} s$ and $\pi_{<\sigma>^\perp} s$ respectively. The statement that σ is $A_\diamond$-covariantly constant follows from the $< \sigma >^\perp$ components of the equations:

$$\begin{aligned} \pi_{<\sigma>^\perp} (d_{A_\diamond} \hat{\varphi}_\diamond) &= 0 \\ \pi_{<\sigma>^\perp} \left(d^*_{A_\diamond} \hat{\varphi}_\diamond\right) &= 0. \end{aligned}$$

3 Data for 3-manifolds & Almgren's Frequency Function

In the case of a Riemann surface, the data Z and $\mathcal{I} \to M - Z$ was defined by the holomorphic quadratic differential. In this section we discuss the definition of Z and $\mathcal{I}$ when the base manifold is 3-dimensional.

3.1 The closed set, Z

Define the suggestively-named non-negative function, $|\hat{\phi}_\diamond|$, at $p \in M$ by

$$|\hat{\phi}_\diamond|(p) := \limsup_{n \to \infty} |\hat{\phi}_n|$$

The normalized Higgs field $\hat{\phi}$ has unit "energy" $|\hat{\phi}|_{L^2(M)}$ and so $|\hat{\phi}|$ can be interpreted as a measure on M of how much energy is concentrating at any given point. Define $Z := |\hat{\phi}_\diamond|^{-1}(0)$, the zero locus of this limiting "energy concentration measure," $|\hat{\phi}_\diamond|$. It is a fact (but not an easy one to prove) that $|\hat{\phi}_\diamond|$ is continuous. The fact that the set, Z, is closed follows from the continuity of $|\hat{\phi}_\diamond|$. We can and will say more about Z.

3.2 A construction of $\mathcal{I}$

Rather than defining $\mathcal{I}$ from the holomorphic quadratic differential, in the 2-manifold case, we morally define $\mathcal{I}$ as the 1-form such that $\sigma\mathcal{I} = \hat{\phi}_\diamond$. More explicitly, we define $\mathcal{I}$ as the bundle dual to the bundle $\mathcal{I}_{\text{dual}} \subset TM$, which is defined as follows: At a fixed point $p \in M$, consider the $\mathbb{R}$-valued map from the unit tangent bundle at p defined by

$$\begin{array}{rcl} S_pM & \to & \mathbb{R}^{\geq 0} \\ \xi & \mapsto & |\hat{\phi}_\diamond(\xi)|^2. \end{array}$$

This map has a "unique" maximum at each point where $|\hat{\phi}_\diamond|(p) \neq 0$, which will turn out to be the set $M - Z$. (Note that if ξ is a maximum then $-\xi$ is also a maximum. These two distinguished points in the unit tangent bundle over $M - Z$ fit into a $\mathbb{Z}_2$-bundle over $M - Z$.) The bundle $\mathcal{I}_{\text{dual}} \to M - Z$ is defined as the real subbundle of $TM|_{M-Z}$ containing the maximum. The bundle $\mathcal{I} \subset T^*M|_{M-Z}$ is the dual of $\mathcal{I}_{\text{dual}}$.

3.3 Almgren's Frequency Function

Almgren's frequency function is an important tool in Taubes' paper. Taubes uses the frequency function to investigate the set Z and describe the topology of the real line bundle $\mathcal{I}$.

In the case of the Riemann surface, the holomorphic quadratic differential did this for us. The set Z was the discrete and finite set of zeros of the holomorphic quadratic differential, $\varphi_{2,\diamond}$. These zeros could be classified by their order. The spectral cover, and hence the bundle $\mathcal{I}$ was locally trivial around zeros of even order. These bundles were non-trivial around zeros of odd order.

In the 3 (and 4)-dimensional cases, there is no handy holomorphic quadratic differential. As a partial and inferior replacement, Taubes introduces a function $N_{(p)}(r)$ which he refers to as the "Almgren's frequency function" (after the Geometric Measure Theorist, Frederick Almgren, Jr.) to investigate the structures of Z and $\mathcal{I}$ near Z. For $r > 0$, the frequency function is defined by [T1, Eq. 5.3]:

$$N_{(p)}(r) = \frac{\int_{B_p(r)} |\nabla\nu|^2}{r^{n-2}\int_{\partial B_p(r)} |\nu|^2} = \frac{r\int_{B_p(r)} \left(|\nabla_A\hat{\phi}|^2 + 2R^2|\hat{\phi}\wedge\hat{\phi}|^2\right)}{\int_{\partial B_p(r)} |\hat{\phi}|^2},$$

where n is the dimension of the base manifold M and $R = ||\phi||_{L^2(M)}$. The frequency function plays a dominant role in §5 − §10 of the first version of Taubes paper.

There are two main uses of the frequency function:

- Extract a number, $N_0(p)$, in order to understand $\mathcal{I}$ near p
- "Zoom in" at p in Z in some "controlled" way, in order to understand the structure of Z.

3.3.1 Classification of points in singular set

The function $N_{(p)}(r)$ is defined for any $p \in M$. It is monotonic non-decreasing [T1, Prop 5.1], consequently, the limit

$$N_{(p)}(0) := \lim_{r\to 0} N_(p)(r)$$

is defined.

$N_{(\cdot)}(0) : M \to \mathbb{R}$ has the following properties:

- $N_{(\cdot)}(0)$ distinguishes between Z and $M - Z$. It it zero on $M - Z$ and positive on Z.
- $N_{(\cdot)}(0)$ is locally constant on Z.
- Though a priori $N_{(\cdot)}(0)$ can be any positive real number on Z, it is instead "quantized." The function $N_{(\cdot)}(0)$ takes values in $\frac{1}{2}\mathbb{Z}^{>0}$ at almost every[2]–and quite likely all–points of Z. Consequently, we can use $N_{(\cdot)}(0)$ to stratify Z into sets. One useful classification of points of Z is into the following disjoint sets:

$$\begin{aligned} Z_{trivial} &= \{p \in Z : N_{(p)}(0) \in \mathbb{Z}^{>0}\} \\ Z_{non-trivial} &= \{p \in Z : N_{(p)}(0) + \frac{1}{2} \in \mathbb{Z}^{>0}\} \\ Z_{strange} &= Z - Z_{trivial} \sqcup Z_{non-trivial}, \end{aligned}$$

[2] In [T2, Lemma 6.6] Taubes proves that $N_{(\cdot)}(0)$ takes values in $\frac{1}{2}\mathbb{Z}^{>0}$ on an open, dense set of Z. Taubes does not prove that $N_{(\cdot)}(0)$ takes values in $\frac{1}{2}\mathbb{Z}^{>0}$ for *all* points in Z, but he appears to suspect this might be the case.

whose names leadingly reflect the following facts:

- The real bundle $\mathcal{I}$ is locally trivial near p if, and only if, $N_p(0)$ is an integer.
- When $\dim X = 2$, $Z_{strange}$ is empty. If $\dim X = 3$, the Hausdorff dimension[3] of $Z_{strange}$ is 0. For completeness, if $\dim X = 4$, the Hausdorff dimension of $Z_{strange}$ is at most one. [T2, Prop 7.1]
- The set $Z_{non-trivial}$, which Taubes calls the "points of discontinuity" is a particularly nice set. It is the closure of an open set in Z and has the structure of a codimension 2, differentiable submanifold in X. [T2, Thm 1.2]

3.3.2 Structure of singular set

One can use the monotonicity of the frequency function to study the structure of Z near p at smaller length scales. One uses the frequency function to "zoom in" in a controlled way at a point p of Z. Then one can consider the tangent cone at p, a common object of study in geometric measure theory.

From this information about the tangent cones, Taubes can get information about the local and global structure of Z. Results about this originally made up §6 − §10 of the first version of Taubes' paper. However, the proof of the original Lemma 8.8 had a hole in it. The problematic lemma claimed that all points of Z have unique tangent cones. While it appears that Taubes suspects this indeed indeed the case, Taubes was able to prove the same big results using a weaker replacement lemma that "enough" points had unique tangent cones.

I won't say any more about the structure of Z because the results are no longer proved in the revised version of Taubes paper; and more importantly, I think most people at the workshop are more interested in the simpler 2-manifold case.

[3]Recall that the Hausdorff dimension of a set is a possibly non-integer value d related to the number of balls of diameter r, $\mathcal{N}(r)$, needed to cover the set as $r \to 0$. The relation is roughly

$$\mathcal{N}(r) \sim r^{-d}.$$

4 Comments on Taubes' Proof

4.1 Why is the metric "harmonic"?

Why, morally, is $d_{A_\diamond}\hat{\phi}_\diamond = 0$ and $d^*_{A_\diamond}\hat{\phi}_\diamond = 0$? Recall that these two equations are the Euler-Lagrange equations of the "energy" of the metric. At first glance, it may be surprising that the limiting configuration would be a "minimizer," in some sense, of the infinite energy, since the sequence (A_n, ϕ_n) need not be minimizers. However, the bound on the curvature functional below means that $(A_n, \hat{\phi}_n)$ aren't "too far" from being minimizers. Let C be the upper bound for the curvature functional

$$\mathcal{F}(A_n + i\phi_n) = \int_M |F_{A_n} - \phi_n \wedge \phi_n|^2 + |d_{A_n}\phi_n|^2 + |d^*_{A_n}\phi_n|^2 \leq C.$$

Then, the normalized Higgs field $\hat{\phi}_n$ satisfies

$$||d_{A_n}\hat{\phi}_n||^2_{L^2(M)} \leq \frac{C}{||\phi_n||^2_{L^2(M)}}$$

As $||\phi_n||^2_{L^2(M)} \to \infty$, the right hand side goes to zero, hence $||d_{A_n}\hat{\phi}_n||_{L^2(M)} \to 0$. Provided one can prove the difficult statement $(A_n, \hat{\phi})$ weakly converges to $(A_\diamond, \hat{\phi}_\diamond)$ in $W^{1,2}_{loc}(M - Z)$, one can show that the limit configuration satisfies $d_{A_\diamond}\hat{\phi}_\diamond = 0$. The argument is similar for $d^*_{A_\diamond}\hat{\phi}_\diamond = 0$.

4.2 Why is $\phi_\diamond \wedge \phi_\diamond = 0$?

Roughly, from the Bochner-Weitzenböck formula, one can show that

$$\{||\hat{\phi}_n \wedge \hat{\phi}_n||_{L^2(M)}\} \to 0.$$

Again, provided one can prove that $(A_n, \hat{\phi}_n)$ converges to $(A_\diamond, \phi_\diamond)$ in the right space, it follows that the limiting configuration satisfies $\hat{\phi}_\diamond \wedge \hat{\phi}_\diamond = 0$.

We derive a bound on $||\hat{\phi}_n \wedge \hat{\phi}_n||_{L^2(M)}$:

$$\begin{aligned}\frac{C_{\mathcal{F}}}{||\phi||^2_{L^2(M)}} &> \int_M \left(|d_{A_n}\hat{\phi}_n|^2 + |d^*_{A_n}\hat{\phi}_n|^2 + ||\phi_n|^{-2}_{L^2(M)}|F_{A_n} - \hat{\phi}_n \wedge \hat{\phi}_n|^2\right) \\ &= \int_x \left(|\nabla_A\hat{\phi}_n|^2 + ||\phi_n||^2_{L^2(M)}|\hat{\phi}_n \wedge \hat{\phi}_n|^2 + ||\phi_n||^{-2}_{L^2(M)}|F_A|^2 + \mathrm{Ric}(\left\langle \hat{\phi}_n \otimes \hat{\phi}_n \right\rangle)\right)\end{aligned}$$

The bound $C_{\mathcal{F}} > \mathcal{F}(A_n, \phi_n)$on the functional gives the first line, and the integral Bochner-Weitzenböck formula with $f = 1$ and $R = ||\phi_n||_{L^2(M)}$ gives the second line. Consequently, moving the Ricci curvature term to the left side, and throwing out some positive terms, we get the following bound

$$\frac{C_{\mathcal{F}}}{||\phi||^2_{L^2(M)}} - \int_M \mathrm{Ric}\left(\left\langle \hat{\phi}_n \otimes \hat{\phi}_n \right\rangle\right) > ||\phi_n||^2_{L^2(M)} ||\hat{\phi}_n \wedge \hat{\phi}_n||^2_{L^2(M)}.$$

There is some constant C_{Ric} such that the possibly-negative Ricci curvature term satisfies $C_{\mathrm{Ric}}||\hat{\phi}_n||^2_{L^2(M)} > -\int_M \mathrm{Ric}\left(\left\langle \hat{\phi}_n \otimes \hat{\phi}_n \right\rangle\right)$, and consequently,

$$\frac{C_{\mathcal{F}}}{||\phi||^2_{L^2(M)}} + C_{Ric} > ||\phi_n||^2_{L^2(M)} ||\hat{\phi}_n \wedge \hat{\phi}_n||^2_{L^2(M)},$$

i.e. $||\hat{\phi}_n \wedge \hat{\phi}_n||_{L^2(M)} \to 0$.

One might hope that one could prove that $||F_{A_n}||_{L^2(M-Z_\varepsilon)}$ is bounded, using the triangle inequality, from

- a bound on $||F_{A_n} - \phi_n \wedge \phi_n||_{L^2(M-Z_\varepsilon)}$ (obtained from the bound on the functional $\mathcal{F}(A_n, \phi_n)$), and
- a bound on $||\phi_n \wedge \phi_n||_{L^2(M-Z_\varepsilon)} = ||\phi_n||^2_{L^2(M)} \cdot ||\hat{\phi}_n \wedge \hat{\phi}_n||_{L^2(M-Z_\varepsilon)}$. *However,* note that we've only demonstrated that $||\phi_n||_{L^2(M)} \cdot ||\hat{\phi}_n \wedge \hat{\phi}_n||_{L^2(M)}$ is bounded.

4.3 Why is $||F_{A_n}||_{L^2(M-Z_\varepsilon)}$ bounded?

What then is Taubes' argument? Let Z_ε be an ε-neighborhood of Z in M.

- If $\{||F_{A_n}||_{L^2(M-Z_\varepsilon)}\}$ is unbounded then there is a a sequence of points $p_n \in M - Z_\varepsilon$ where curvature concentrates in balls of smaller and smaller radius ρ'_n. [Reference: [T1] Lemma 7.2]
- Since $M - Z_\varepsilon$ is compact, there is a subsequence p_n converging to some p. Because curvature is concentrating in balls of decreasing radius ρ'_n centered at p_n, curvature is also concentrating in balls of decreasing radius ρ_n centered at p.
- However, curvature is not allowed to concentrate at p (particularly, $\{\rho_n(p)\}$ is bounded below) if $|\hat{\phi}_\diamond|(p) > 0$. One can prove that if the quantized Almgren's frequency $N_{(\cdot)}(0) : M \to \mathbb{R}$ vanishes, then $|\hat{\phi}_\diamond|(p) > 0$. (We briefly discussed in Section 3.3.1 that $N_{(\cdot)}(0)$ distinguishes between Z and $M - Z$ by vanishing on $M - Z$.) Consequently, curvature can't concentrate at any point in $M - Z_\varepsilon$, and so it is bounded on $M - Z_\varepsilon$. [Reference: [T1] Lemma 6.2.]

Note that in Taubes' paper, the proof that curvature is bounded away from Z, the proof that $|\hat{\phi}_\diamond|$ is continuous, and the proof that $\{(A_n, \hat{\phi}_n)\}_n$ convergences to $(A_\diamond, \phi_\diamond)$ in the right Sobolev space are all inextricably intertwined. I've somewhat misrepresented Taubes' proof in assuming that these can be separated, and I've completely ignored the very important Proposition 3.2. However, I hope that this simplification is forgivable in a survey talk.

Once we have a bound on curvature, we can use Uhlenbeck compactness to extract a subsequence and a corresponding sequence of gauge transformations $\{g_n\}$ on $M - Z$ such that

$$g_n^* A_n \rightharpoonup A_\diamond.$$

It still remains to show that there is a subsubsequence such that

$$g_n^* \hat{\phi}_n \rightharpoonup \hat{\phi}_\diamond.$$

4.4 Comments on convergence

In the sections above, we've discussed why the sequences $\{d_{A_n}\hat{\phi}_n\}$, $\{d^*_{A_n}\hat{\phi}_n\}$, and $\{\hat{\phi}_n \wedge \hat{\phi}_n\}$ all approach 0. However, without proving that $(A_n, \hat{\phi}_n)$ weakly converges to $(A_\diamond, \hat{\phi}_\diamond)$ in $W^{2,1}_{loc}(M - Z)$, we can't say that the limiting configuration satisfies $d_{A_\diamond}\hat{\phi}_\diamond = 0$, $d^*_{A_\diamond}\hat{\phi}_\diamond = 0$, and $\hat{\phi}_\diamond \wedge \hat{\phi}_\diamond = 0$. I won't say anything about the proof of this convergence. However, Taubes actually proves that there is a subsequence such that $\{g_n^*\hat{\phi}_n\}$ weakly converges to $\hat{\phi}_\diamond$ in the $W^{2,2}_{loc}(M-Z)$-topology. By elliptic regularity, Taubes get bounds on all second derivatives, packaged together as $\nabla_A \nabla_A \hat{\phi}$, in terms of the particular second derivative $(d_A d_A^* + d_A^* d_A)\hat{\phi}$. Back at the beginning of the talk we said the $(d_A + d_A^*)$ and its square $(d_A d_A^* + d_A^* d_A)$ were important! We'll see the Euler-Langrange equations of the "energy" one last time before the end of the talk.

At a number of places in the proof, Taubes "regularizes" the sequence. $(A_\diamond, \hat{\phi}_\diamond)$ is the limit of the sequence $(A_n, \hat{\phi}_n)$. However, in order to prove that the limiting configuration $(A_\diamond, \hat{\phi}_\diamond)$ has nice properties, Taubes must modify the initial sequence to a nearby nicer sequence. What properties do we want this nicer sequence to have?

- $\{||\hat{\phi}_n||_{L^\infty(M)}\}$ is uniformly bounded. (Taubes uses this in the proof that $|\hat{\phi}_\diamond|$ is continuous) Note: Remember that $|\hat{\phi}_n|$ vanishes –rather than blows up–on Z.

- $||(d_{A_n}d^*_{A_n} + d^*_{A_n}d_{A_n})\hat{\phi}_n||_{L^2(M)}$ is uniformly bounded. (By elliptic regularity, Taubes uses this to get uniform bounds on $\nabla_{A_n}\nabla_{A_n}\phi_n$, and then proves that there is a *sub*subsequence such that $g^*_n\hat{\phi}_n$ converges to $\hat{\phi}_\diamond$.)

The way to get such a sequence is to flow our initial sequence of Higgs fields $\{\phi_n\}$ to a new sequence $\{\phi'_n\}$. Let ϕ_t be a solution of

$$\begin{aligned} \frac{d}{dt}\phi_t &= -(d^*_A d_A + d_A d^*_A)\phi_t \\ \phi_0 &= \phi. \end{aligned}$$

This flow is nice and there is some small T so that if we can replace $\{\phi_n = (\phi_n)_0\}$ with the sequence $\{(\phi_n)_T\}$, we have the sort of "nice" sequence discussed above. One can interpret this flow as adjusting ϕ so our fixed intial hermitian metric is closer to a harmonic metric.

5 References

[T1] C. H. Taubes "$PSL(2,\mathbb{C})$ connections on 3-manifolds," (2012/2014) arXiv:1205.0514.

[T1] C. H. Taubes "The zero loci of $\mathbb{Z}/2$ harmonic spinors in dimensions 2,3,4" (2014) arXiv:1407.6206.