Topological invariants for Anosov representations

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Abstract Our goal in these notes will be to define topological invariants for Anosov representations into $\mathrm{Sp}(2n,\mathbb{R})$, concentrating particularly on the $\mathrm{Sp}(4,\mathbb{R})$ case, which allow us to distinguish connected components of the set of maximal representations. We will also describe how to construct model representations lying in those connected components. These notes are based on [1], all of the results are taken directly from that paper.

1 The setting

In this introductory section, we describe the setting of Anosov and maximal representations. The invariants described later on will be characteristic classes of bundles associated to the representations, so in light of different equivalent definitions of Anosov representations that were given recently, we begin by fixing the definition we use and the general setup.

1.1 Anosov representations

Let M be a compact manifold equipped with an Anosov flow ϕ_t, G a connected semisimple Lie group, (P^+,P^-) a pair of opposite parabolic subgroups with intersection $H = P^+ \cap P^-$, and $\mathscr{F}^\pm = G/P^\pm$ the associated flag varieties.
We let $\mathscr{X}$ be the open G-orbit in $\mathscr{F}^+ \times \mathscr{F}^-$. It is foliated according to this product structure, and we denote the corresponding G-invariant distributions by E^+ and E^-.

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Definition 1. A flat G-bundle $P \to M$ is a (G,H)-*Anosov bundle* if:

1. P admits an H-reduction that is flat along low lines, i.e.:
   - There exists a section $\sigma : M \to P \times_G \mathscr{X}$.
   - σ is flat (locally constant with respect to the flat structure) along flow lines of ϕ_t.
2. Letting $E^{\pm}$ also denote the induced distributions on $P \times_G \mathscr{X}$ and ϕ_t the lift of the flow (using the flat connection), this flow is contracting on $\sigma^* E^+$ and dilating on $\sigma^* E^-$.

Such a section σ is called an *Anosov section* or *Anosov reduction*.

Definition 2. A representation $\rho : \pi_1(M) \to G$ is (G,H)-*Anosov* or H-*Anosov* if the flat G-bundle $\tilde{M} \times_{\pi_1(M)} G$ is a (G,H)-Anosov bundle.

If there is an invariant volume form for the flow ϕ_t, the Anosov section is unique. This assumption is satisfied in the surface case to which we will specialize soon, and will be assumed to be true from now on. To emphasize this statement, we put it in form of a proposition:

Proposition 1. *The Anosov section for an Anosov bundle $P \to M$ is unique. In particular, a (G,H)-Anosov bundle admits a canonical H-reduction.*

To deduce later on that certain invariants are indeed constant on connected components of Anosov representations, we will also need the following result:

Proposition 2. *The set of (G,H)-Anosov representations is open in the set of all representations* $\mathrm{Hom}(\pi_1(M), G)$. *The Anosov section σ depends continuously on the representation.*

We now restrict our attention to the case of closed oriented connected surfaces Σ of negative Euler characteristic. Then $M = T^1\Sigma$ and ϕ_t is the geodesic flow for some hyperbolic metric.

Definition 3. A flat G-bundle P over Σ is *Anosov* if the pullback bundle $\pi^* P$ over $T^1\Sigma$ is Anosov.
A representation $\rho : \pi_1(\Sigma) \to G$ is *Anosov* if $\pi_1(T^1\Sigma) \to \pi_1(\Sigma) \to G$ is Anosov.

Identifying the unit tangent bundle of the universal cover, $T^1\tilde{\Sigma}$, with $\partial\pi_1(\Sigma)^{(3+)}$ (the set of distinct positively oriented triples), yields the two ρ-equivariant boundary maps $\xi^{\pm} : \partial\pi_1(\Sigma) \to \mathscr{F}^{\pm}$.

1.2 Maximal representations

In the following constructions, we will also make use of maximal representations, which are representations with maximal Toledo invariant (see the earlier talk by

Beatrice Pozzetti). Since they are defined by a characteristic class taking its maximal value, they consist of a union of connected components of the space of all representations. We now recall some facts about maximal representations which will be useful later. First of all, every maximal representation is Anosov:

Theorem 1. *Let $\rho : \pi_1(\Sigma) \to G$ be a maximal representation. Then ρ is (G,H)-Anosov, where H is the stabilizer of a pair of transverse points in the Shilov boundary of the symmetric space associated with G. For $G = \mathrm{Sp}(2n,\mathbb{R})$, H is the stabilizer of a pair of transverse Lagrangians in $\mathbb{R}^{2n}$.*

Another useful fact is the following gluing theorem which allows to construct new representations by combining several representations.

Theorem 2. *Let $\Sigma = \Sigma_1 \cup_\gamma \Sigma_2$ be decomposed along the simple closed curve γ, $\pi_1(\Sigma) = \pi_1(\Sigma_1) *_{\langle\gamma\rangle} \pi_1(\Sigma_2)$. Let $\rho_i : \pi_1(\Sigma_i)$ be representations such that $\rho_1(\gamma) = \rho_2(\gamma)$, and $\rho = \rho_1 * \rho_2 : \pi_1(\Sigma) \to G$ the amalgamated representation. Then we have: ρ is maximal if and only if ρ_1 and ρ_2 are maximal.*

There is another viewpoint on the H-reduction which will be used to define some invariants. We will describe it somewhat explicitly for $\mathrm{Sp}(2n,\mathbb{R})$, so let ω be the symplectic form given by $\begin{pmatrix} 0 & \mathrm{Id}_n \\ -\mathrm{Id}_n & 0 \end{pmatrix}$ and $P^\pm$ be the stabilizers of the transverse Lagrangians $\mathbb{R}^n \times \{0\}$ and $\{0\} \times \mathbb{R}^n$. Their intersection H is isomorphic to $\mathrm{GL}(n,\mathbb{R})$ via

$$A \mapsto \begin{pmatrix} A & 0 \\ 0 & (A^T)^{-1} \end{pmatrix}$$

Now let $\rho : \pi_1(\Sigma) \to \mathrm{Sp}(2n,\mathbb{R})$ be maximal and therefore $(\mathrm{Sp}(2n,\mathbb{R}), \mathrm{GL}(n,\mathbb{R}))$-Anosov, P be the flat $\mathrm{Sp}(2n,\mathbb{R})$-bundle over $T^1\Sigma$ and E the associated $\mathbb{R}^{2n}$-bundle. The Anosov $\mathrm{GL}(n,\mathbb{R})$-reduction is equivalent to a flow-invariant, continuous splitting of E into two transverse Lagrangian subbundles, $E = L^+(\rho) \oplus L^-(\rho)$. It is connected to the boundary map $\xi : \partial\pi_1(\Sigma) \to \mathscr{L}$ mapping into the space of Lagrangians as follows: The fiber $L^+(\rho)_v$ over a point $v \in T^1\Sigma$ is given by $\xi(t^+)$, where a lift of v to $T^1\tilde{\Sigma}$ is identified with $(t^+,t,t^-) \in \partial\pi_1(\Sigma)^{(3+)}$ (by ρ-equivariance, the choice of lift does not matter).
Since $L^-(\rho) \cong L^+(\rho)^*$ via the symplectic form, we will use only $L^+(\rho)$ to define the invariants.

For maximal representations, the limit curve $\xi : \pi_1(\Sigma) \to \mathscr{L}$ into the space of Lagrangians has the additional property of being a positive curve (see the previous talk on maximal representations).

2 Model representations

We now describe the different model representations we will use to distinguish connected components. They are all maximal representations and can therefore be used in gluing constructions.

2.1 Irreducible Fuchsian representations

First of all, we have the irreducible Fuchsian representations. Those are representations obtained by composing a discrete embedding $\iota : \pi_1(\Sigma) \to \mathrm{SL}(2,\mathbb{R})$ with the irreducible representation $\mathrm{SL}(2,\mathbb{R}) \to \mathrm{Sp}(2n,\mathbb{R})$:

$$\rho_{irr} : \pi_1(\Sigma) \to \mathrm{SL}(2,\mathbb{R}) \to \mathrm{Sp}(2n,\mathbb{R})$$

Let E_ι and $E_{\rho_{irr}}$ be the associated flat $\mathbb{R}^2$ and $\mathbb{R}^{2n}$ bundles. Then by definition of the irreducible representation we have $E_{\rho_{irr}} = \mathrm{Sym}^{2n-1} E_\iota$, and since we also have the splitting $E_\iota = L^+(\iota) \oplus L^-(\iota) = L^+(\iota) \oplus L^+(\iota)^* = L^+(\iota) \oplus L^+(\iota)^{-1}$ into lines bundles, this yields

$$L^+(\rho_{irr}) = L^+(\iota)^{2n-1} \oplus L^+(\iota)^{2n-3} \oplus \ldots \oplus L^+(\iota).$$

2.2 Diagonal Fuchsian representations

Next, there are the diagonal Fuchsian representations. To define those, split $\mathbb{R}^{2n} = W_1 \oplus \ldots \oplus W_n$ into 2-dimensional symplectic subspaces. Then a diagonal Fuchsian representation ρ_Δ is given by

$$\pi_1(\Sigma) \xrightarrow{\iota} \mathrm{SL}(2,\mathbb{R}) \xrightarrow{\mathrm{diag}} \mathrm{SL}(2,\mathbb{R})^n \xrightarrow{\psi} \mathrm{Sp}(2n,\mathbb{R}),$$

where the last embedding is induced by the splitting.
In this case, the Lagrangian bundle over $T^1\Sigma$ is given by

$$L^+(\rho_\Delta) = L^+(\iota) \oplus \ldots \oplus L^+(\iota).$$

2.3 Twisted diagonal representations

A variation on the diagonal representations are twisted diagonal representations. To define them, we use a different viewpoint on the diagonal Fuchsian representations. Let $V = \mathbb{R}^2$ be equipped with the standard symplectic form ω_0, and W be an n-dimensional vector space carrying a definite quadratic form q. Then $V \otimes W$ inherits the symplectic form $\omega_0 \otimes q$, and this identification induces an embedding

$$\mathrm{SL}(2,\mathbb{R}) \times \mathrm{O}(n) \xrightarrow{\phi_\Delta} \mathrm{Sp}(2n,\mathbb{R}).$$

This embedding extends the diagonal embedding of $\mathrm{SL}(2,\mathbb{R})$, and the image of $\mathrm{O}(n)$ is exactly the centralizer of the image of $\mathrm{SL}(2,\mathbb{R})$ (the centralizer of a maximal representation fixes the positive boundary curve, and the stabilizer of any pos-

itive triple of Lagrangians is isomorphic to $\mathrm{O}(n)$, so it cannot be bigger than this embedded $\mathrm{O}(n)$).
So given $\iota : \pi_1(\Sigma) \to \mathrm{SL}(2,\mathbb{R})$ and $\Theta : \pi_1(\Sigma) \to \mathrm{O}(n)$, we define the twisted diagonal representation

$$\begin{aligned} \rho_\Theta = \iota \otimes \Theta : \pi_1(\Sigma) &\to \mathrm{Sp}(2n,\mathbb{R}) \\ \gamma &\mapsto \phi_\Delta(\iota(\gamma),\Theta(\gamma)) \end{aligned}$$

which is given by multiplication with a representation into the centralizer of the diagonal embedding of $\mathrm{SL}(2,\mathbb{R})$.
The flat vector bundle associated with ρ_Θ has the form $E_\iota \otimes E_\Theta$, where E_ι is the flat bundle associated with ι and E_Θ is the flat bundle associated with Θ. Similarly, letting $\bar{E}_\Theta$ denote the pullback bundle over $T^1\Sigma$, the Lagrangian reduction of E_{ρ_Θ} is given by $L^+(\rho_\Theta) = L^+(\iota) \otimes \bar{E}_\Theta$.

2.4 Hybrid representations

Now we come to the gluing construction foreshadowed by theorem 2. If $\Sigma = \Sigma_l \cup_\gamma \Sigma_r$ is a decomposition along a simple closed separating oriented geodesic, and $\rho_l : \pi_1(\Sigma_l) \to \mathrm{Sp}(4,\mathbb{R}), \rho_r : \pi_1(\Sigma_r) \to \mathrm{Sp}(4,\mathbb{R})$ are maximal representations which agree on γ, we can consider the maximal representation $\rho = \rho_l * \rho_r : \pi_1(\Sigma) \to \mathrm{Sp}(4,\mathbb{R})$. The idea is to take an irreducible Fuchsian representation on one of the two subsurfaces and a diagonal Fuchsian representation on the other. A slight modification is necessary to make them agree on γ.
We skip the details of this modification (which can be found in [1]) and just give an outline: Instead of using the diagonal embedding $\phi_\Delta \circ \iota$ for ρ_r, we need to modify the two components: Letting $(\iota_{1,t}, \iota_{2,t})$ be a path starting at $t = 0$ at (ι,ι), we deform the two discrete embeddings $\pi_1(\Sigma) \to \mathrm{SL}(2,\mathbb{R})$ so that $\rho_l(\gamma) = \rho_r(\gamma)$, where $\rho_r = \psi \circ (\iota_{1,1}, \iota_{2,1})$ is a deformation of $\phi_\Delta \circ \iota$.

Definition 4. A *k-hybrid representation* is a representation obtained as the amalgamated product described above, where $\chi(\Sigma_l) = k$.

This construction can be generalized to allow for decompositions of Σ which have a tree as dual graph, see [1]

3 Topological invariants

In this chapter, we describe how to associate the topological invariants to representations. First of all, continuity and uniqueness of the Anosov section implies the following proposition. We write $\mathscr{B}_H(M)$ for the set of gauge isomorphism classes of H-bundles over M.

Proposition 3. *The map* $\mathrm{Hom}_{H-\mathrm{Anosov}}(\pi_1(\Sigma),G) \to \mathscr{B}_H(T^1\Sigma)$ *sending an Anosov representation to its H-reduction is well-defined and locally constant.*

3.1 Stiefel-Whitney-classes

As a consequence, we can associate to a connected component of Anosov representations some characteristic classes of the associated H-reduction: To any $\rho \in \mathrm{Hom}_{\max}(\pi_1(\Sigma),\mathrm{Sp}(2n,\mathbb{R}))$, we associate the first and second Stiefel-Whitney classes of the associated $\mathrm{GL}(n,\mathbb{R})$ bundle over $T^1\Sigma$, giving us maps

$$\mathrm{Hom}_{\max}(\pi_1(\Sigma),\mathrm{Sp}(2n,\mathbb{R})) \xrightarrow{sw_1} \mathrm{H}^1(T^1\Sigma;\mathbb{Z}/2\mathbb{Z})$$
$$\mathrm{Hom}_{\max}(\pi_1(\Sigma),\mathrm{Sp}(2n,\mathbb{R})) \xrightarrow{sw_2} \mathrm{H}^2(T^1\Sigma;\mathbb{Z}/2\mathbb{Z}).$$

3.2 Euler class

In the $\mathrm{Sp}(4,\mathbb{R})$ case, the first and second Stiefel-Whitney classes are not sufficient to distinguish connected components and we also associate an Euler class to representations. Whenever $sw_1(\rho)=0$, the Lagrangian bundle $L^+(\rho)$ is orientable, but it does not come with a preferred orientation. We therefore denote y $\mathscr{L}_+$ the space of oriented Lagrangians, $\pi : \mathscr{L}_+ \to \mathscr{L}$ the projection and introduce the space

$$\mathrm{Hom}^{\mathscr{L}_+,\gamma}_{\max,sw_1=0}(\pi_1(\Sigma),\mathrm{Sp}(4,\mathbb{R})) \subset \mathrm{Hom}_{\max,sw_1=0}(\pi_1(\Sigma),\mathrm{Sp}(4,\mathbb{R})) \times \mathscr{L}_+$$

of pairs (ρ,L_+) with $\pi(L_+)=L^+(\gamma)$. The reason for doing so is that the limit curve can be lifted to a curve of oriented Lagrangians, and by fixing one specific oriented Lagrangian, we fix the lift.

Proposition 4. *Let* $\rho \in \mathrm{Hom}_{\max}(\pi_1(\Sigma),\mathrm{Sp}(4,\mathbb{R}))$ *with limit curve* $\xi : \partial\pi_1(\Sigma) \to \mathscr{L}$. *Then* ξ *can be lifted to* $\mathscr{L}_+$ *and the lift is equivariant if and only if* $sw_1(\rho)=0$.

As mentioned, we can therefore define:

Definition 5. Let $\rho \in \mathrm{Hom}^{\mathscr{L}_+,\gamma}_{\max,sw_1=0}(\pi_1(\Sigma),\mathrm{Sp}(4,\mathbb{R}))$. Then the lift ξ_+ of ξ with $\xi_+(\gamma_+)=L_+$ is called the *canonical oriented equivariant curve* for the pair (ρ,L_+). Here we denoted by γ_+ the attractive fixed point of γ.

This lift induces a splitting $E = L^+_{or}(\rho) \oplus L^-_{or}(\rho)$ into two oriented Lagrangian subbundles.

Definition 6. The Euler class of a pair (ρ,L_+) is the Euler class of the $\mathrm{GL}^+(2,\mathbb{R})$ reduction.

For a representation $\rho \in \mathrm{Hom}_{\max,sw_1=0}(\pi_1(\Sigma),\mathrm{Sp}(4,\mathbb{R}))$ and fixed $\gamma \in \pi_1(\Sigma)\setminus\{0\}$, $L_+ \in \mathscr{L}_+$, we define the *relative Euler class* $e_{\gamma,L_+}(\rho) \in \mathrm{H}^2(T^1\Sigma;\mathbb{Z})$ as the

Euler class of a conjugate ρ' of ρ with $\pi(L_+)$ the attracting Lagrangian of $\rho(\gamma)$. We note as a fact that the choice involved does not matter as this set of conjugates is connected.

3.3 Restrictions on the invariants

There are certain restraints on the possible values these invariants can take, as could be expected since all of the bundles used in the construction are flat along the flow direction.

Proposition 5. *Let* $\rho \in \mathrm{Hom}_{\max}(\pi_1(\Sigma), \mathrm{Sp}(2n,\mathbb{R}))$. *Then:*

1. *For n even,* $sw_1(\rho)$ *lies in the image of* $\mathrm{H}^1(\Sigma;\mathbb{Z}/2\mathbb{Z})$ *in* $\mathrm{H}^1(T^1\Sigma;\mathbb{Z}/2\mathbb{Z})$. *For n odd,* $sw_1(\rho)$ *lies in the (unique) other coset of* $\mathrm{H}^1(\Sigma;\mathbb{Z}/2\mathbb{Z})$.
2. $sw_2(\rho)$ *lies in the image of* $\mathrm{H}^2(\Sigma;\mathbb{Z}/2\mathbb{Z})$ *in* $\mathrm{H}^2(T^1\Sigma;\mathbb{Z}/2\mathbb{Z})$.
3. *For* $\rho \in \mathrm{Hom}_{\max, sw_1=0}(\pi_1(\Sigma), \mathrm{Sp}(4,\mathbb{R}))$, $e_{\gamma,L_+}(\rho)$ *lies in the image of* $\mathrm{H}^2(\Sigma;\mathbb{Z})$ *in* $\mathrm{H}^2(T^1\Sigma;\mathbb{Z})$.

4 Computations of invariants and applications

In this last section, we shortly summarize some results of the computation of the invariants for the model representations introduced earlier. We also state some theorems that follow from these computations.

4.1 Irreducible Fuchsian representations

For irreducible Fuchsian representations into $\mathrm{Sp}(2n,\mathbb{R})$, we saw earlier that

$$L^+(\rho_{irr}) = L^+(\iota)^{2n-1} \oplus L^+(\iota)^{2n-3} \oplus \ldots \oplus L^+(\iota),$$

so, using the properties of Stiefel-Whitney-classes, we obtain

$$\begin{aligned} sw_1(\rho_{irr}) &= n sw_1(L^+(\iota)) \\ sw_2(\rho_{irr}) &= \frac{n(n-1)}{2}(g-1) \bmod 2. \end{aligned}$$

4.2 Diagonal Fuchsian representations

For diagonal Fuchsian representations into $\mathrm{Sp}(2n,\mathbb{R})$, we saw that

$$L^+(\rho_\Delta) = L^+(\iota) \oplus \ldots \oplus L^+(\iota),$$

so we obtain

$$sw_1(\rho_\Delta) = n sw_1(L^+(\iota))$$
$$sw_2(\rho_\Delta) = \frac{n(n-1)}{2}(g-1) \bmod 2.$$

4.3 Twisted diagonal representations

For twisted diagonal representations, the main result is as follows:

Proposition 6. *Let* $(\alpha,\beta) \in \mathrm{H}^1(T^1\Sigma;\mathbb{Z}/2\mathbb{Z}) \times \mathrm{H}^2(T^1\Sigma;\mathbb{Z}/2\mathbb{Z})$ *be a pair satisfying the constraints from proposition 5. In the case* $n = 2$, *we assume that* $\beta = (g-1) \bmod 2$.
Then there exists a twisted diagonal representation ρ_Θ *into* $\mathrm{Sp}(2n,\mathbb{R})$ *with* $sw_1(\rho_\Theta) = \alpha$ *and* $sw_2(\rho_\Theta) = \beta$.

Similar to the previous cases, we saw that

$$L^+(\rho_\Theta) = L^+(\iota) \otimes \bar{E}_\Theta,$$

and we have that $sw_i(\bar{E}_\Theta) = \pi^* sw_i(E_\Theta)$. Therefore, one needs to study the Stiefel-Whitney classes of orthogonal representations $\Theta : \pi_1(\Sigma) \to \mathrm{O}(n)$. However, we do not carry out this analysis here and refer the reader to [1].

We do, however, state a result of this computation:

Theorem 3. *For* $n \geq 3m$ *the map*

$$\mathrm{Hom}_{\max}(\pi_1(\Sigma), \mathrm{Sp}(2n,\mathbb{R})) \setminus \mathrm{Hom}_{\mathrm{Hitchin}}(\pi_1(\Sigma), \mathrm{Sp}(2n,\mathbb{R})) \xrightarrow{sw_1, sw_2} \mathrm{H}^1(T^1\Sigma;\mathbb{Z}/2\mathbb{Z}) \times \mathrm{H}^2(T^1\Sigma;\mathbb{Z}/2\mathbb{Z})$$

induces a bijection from connected components to the set of pairs satisfying the restrictions from Proposition 5.

This theorem follows from the previous proposition, the observation that twisted diagonal representations are never Hitchin since they are not purely loxodromic, and the previously known result that the total number of connected components is $3 \cdot 2^{2g}$ ([2]) (since the number of possibilities in the proposition is $2 \cdot 2^{2g}$). Each of the non-Hitchin components contains a twisted diagonal representation.

4.4 Hybrid representations

For hybrid representations, the computations are more elaborate, so again we will only state some results.

Theorem 4. *For any fixed* $\gamma \in \pi_1(\Sigma) \setminus \{1\}$, $L_+ \in \mathscr{L}_+$, *the relative Euler class* e_{γ,L_+} *distinguishes connected components of*

$$\mathrm{Hom}_{\max,sw_1=0}(\pi_1(\Sigma), \mathrm{Sp}(4,\mathbb{R})) \setminus \mathrm{Hom}_{\mathrm{Hitchin}}(\pi_1(\Sigma), \mathrm{Sp}(4,\mathbb{R})).$$

If $g-1 \neq l \in \mathbb{Z}/(2g-2)\mathbb{Z} \cong \mathrm{H}^2(T^1\Sigma;\mathbb{Z})^{tor}$, *the preimage* $e^{-1}_{\gamma,L_+}(l)$ *is a connected component which contains a k-hybrid representation, for* $k = g-1-l \bmod 2g-2$.

Theorem 5. *Any representation in* $\mathrm{Hom}_{\max,sw_1=0}(\pi_1(\Sigma), \mathrm{Sp}(4,\mathbb{R}))$ *with Euler class not equal to* $(g-1)[\Sigma]$ *has Zariski dense image in* $\mathrm{Sp}(4,\mathbb{R})$.

The proof of this theorem proceeds by inspecting the different possibilities for the Zariski closure of the image of ρ. As it turns out, for any option other than the Zariski closure being all of $\mathrm{Sp}(4,\mathbb{R})$, The Euler class is necessarily equal to $g-1$.

References

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