Chapter 1
Geometrization of Hitchin representations in $\mathbf{PSL}(4,\mathbb{R})$

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Let Σ be a closed oriented surface of genus $g > 1$ and G a real split simple Lie group (for example $\mathrm{PSL}(n,\mathbb{R}),\ \mathrm{PSO}(n,n),\ \mathrm{PSO}(n,n+1),\ \mathrm{PSp}(2n,\mathbb{R})$). It follows from the work of Kostant [**?**] that there exists a unique (up to conjugation) irreducible morphism

$$\rho_{\mathrm{G}} : \mathrm{PSL}(2,\mathbb{R}) \longrightarrow \mathrm{G}.$$

This morphism provides a preferred embedding ι_G of the Teichmüller space $\mathscr{T}(\Sigma)$ into the moduli space $\mathscr{R}(\Sigma,\mathrm{G}) := \mathrm{Hom}^{\mathrm{ss}}(\pi_1\Sigma,\mathrm{G})/\mathrm{G}$ of conjugacy classes of semi-simple morphisms from the fundamental group of Σ into G.

Using the theory of Higgs bundles, N. Hitchin proved the following result [**?**]

Theorem 1 (Hitchin 1992). *The connected component of $\mathscr{R}(\Sigma,G)$ containing $\iota_G(\mathscr{T}(\Sigma))$ is homeomorphic to a ball of dimension* $\dim(G)(2g-2)$.

This preferred component is now called the *Hitchin component* and will be denoted $\mathscr{H}(\Sigma,G)$. However, as pointed out by N. Hitchin [**?**]: *"Unfortunately, the analytical point of view used for the proofs gives no indication of the geometrical significance of the Teichmüller component."*

The goal is to interpret the representations in the Hitchin component for $\mathrm{PSL}(4,\mathbb{R})$ as holonomy of projective structures.

Definition 1. A projective structure on a n-dimensional manifold M is an atlas of charts taking value in open set of $\mathbb{RP}^n$ such that the transition functions are restriction of elements in $\mathrm{PGL}(n+1,\mathbb{R})$. Two projective structures on M are equivalent if there exists a homeomorphism $h : M \longrightarrow M$ isotopic to the identity whose expression in local charts is given by projective transformations.

We denote by $\mathscr{P}(M)$ the space of equivalence classes of projective structures on M.

Given an equivalence class of projective structures on M, one can associate a developing pair $(\mathrm{dev},\mathrm{hol})$ where

$$\begin{cases} \text{dev} : \widetilde{M} \longrightarrow \mathbb{RP}^n \\ \text{hol} : \pi_1 M \longrightarrow \text{PGL}(n+1,\mathbb{R}), \end{cases}$$

and dev is hol-equivariant and locally injective. Moreover, two developing pairs $(\text{dev}_1, \text{hol}_1)$ and $(\text{dev}_2, \text{hol}_2)$ correspond to the same projective structure if and only if there exists an element $g \in \text{PGL}(n+1,\mathbb{R})$ and a homeomorphism $h : M \longrightarrow M$ isotopic to the identity such that

$$\begin{cases} \text{dev}_1 \circ \widetilde{h} = g^{-1} \circ \text{dev}_2 \\ \text{hol}_2(\gamma) = g \circ \text{hol}_1 \circ g^{-1}, \ \forall \gamma \in \pi_1 M. \end{cases}$$

In particular, we get a well-defined map

$$\textbf{hol} : \mathscr{P}(M) \longrightarrow \text{Hom}(\pi_1 M, \text{PGL}(n+1,\mathbb{R})/\sim .$$

From now on, $M = T^1\Sigma$, the unit tangent bundle of Σ. In these notes we will explain the following result [**?**]:

Theorem 2 (Guichard, Wienhard). $\mathscr{H}(\Sigma, PSL(4,\mathbb{R}))$ *naturally parametrizes the space of properly convex foliated (pcf) projective structures on M. More precisely, the holonomy of a pcf projective structure on M factors through a Hitchin representation* $\rho : \pi_1\Sigma \longrightarrow PSL(4,\mathbb{R})$*. Moreover, each Hitchin representation is the holonomy of a unique pcf-projective structure on M.*

In the first section, we describe the geometry of M and define the moduli space $\mathscr{P}_{pcf}(M)$ of pcf-projective structures on M. In section 2, we give some examples of projective structures on M. In Section 3, we associate a pcf-projective structure on M associated to a Hitchin representation. Finally in Section 4, we explain the geometric structures associated to Hitchin representations in $\text{PSp}(4,\mathbb{R})$.

1.1 The geometry of M

Denote by $\Gamma := \pi_1\Sigma$, and $\overline{\Gamma} = \pi_1 M$. We have the following exact sequence:

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\iota} \overline{\Gamma} \xrightarrow{\pi_*} \Gamma \longrightarrow 1,$$

associated to the covering

$$\widetilde{M} \xrightarrow{\mathbb{Z}} \overline{M} := T^1\widetilde{\Sigma} \xrightarrow{\Gamma} M.$$

Set $\tau := \iota(1) \in \overline{\Gamma}$. We have the following presentations

$$\Gamma = \left\langle a_1, b_1, ..., a_g, b_g | [a_1, g_1]...[a_g, b_g] = 1 \right\rangle$$

$$\overline{\Gamma} = \left\langle a_1, b_1, ..., a_g, b_g, \tau | [a_1, g_1]...[a_g, b_g] = \tau^{2g-2} \right\rangle.$$

Fix a hyperbolic metric h on Σ. Such a metric provides an equivariant identification between $\widetilde{\Sigma}$ and the hyperbolic disk $\mathbb{H}^2$ and an identification between $\overline{M}$ and $T^1\mathbb{H}^2$.

The PSL$(2,\mathbb{R})$ model: The group PSL$(2,\mathbb{R})$ acts freely and transitively on $T^1\mathbb{H}^2$, so fixing a unit tangent vector $u \in T^1\mathbb{H}^2$ provides an identifcation between PSL$(2,\mathbb{R})$ and $T^1\mathbb{H}^2$ (given by the orbit map).

The triple of points model: A point $(p,u) \in T^1\mathbb{H}^2$ defines canonically a triple of positively oriented points (x_-, x_t, x_+) on the boundary $\partial\mathbb{H}^2$ in the following way: the geodesic $\gamma \subset \mathbb{H}^2$ tangents to v intersects $\partial\mathbb{H}^2$ in x_- and x_+. Moreover, there exists a unique geodesic intersecting γ orthogonally at p; set x_t the intersection of this geodesic with $\partial\mathbb{H}^2$ such that (x_-, x_t, x_+) is positively oriented (see Figure 1.1). We get a canonical identification between $\overline{M}$ and the space $\partial\mathbb{H}^{2^{(3)}}$ of positively oriented pairwise distinct triple of points on $\partial\mathbb{H}^2$.

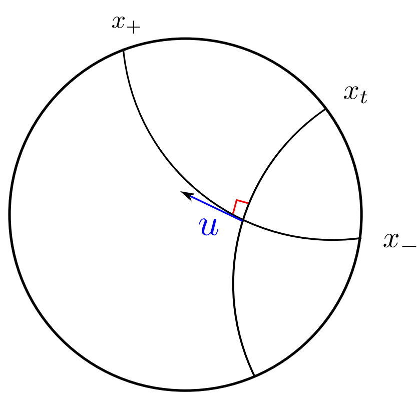

Fig. 1.1 The unit tangent bundle to $\mathbb{H}^2$.

It is well-known (see for instance [**?**]) that the geodesic flow φ on $\overline{M}$ is Anosov as so defines two foliations:

- A codimension 2 foliation by geodesic leaves, where the geodesic leaf $\overline{g}(x)$ passing through $x \in \overline{M}$ is
$$\overline{g}(x) = \{\varphi_t(x),\ t \in \mathbb{R}\}.$$
We denote by $\overline{\mathscr{G}}$ (respectively $\widetilde{\mathscr{G}}$) the space of geodesic leaves on M (respectively the space of lift of geodesic leaves on $\widetilde{M}$).
- A codimension 1 foliation by (weakly) stable leaves, where the stable leaf $\overline{f}(x)$ passing through $x \in \overline{M}$ is
$$\overline{f}(x) = \{y \in \overline{M},\ \big(d_M\big(\varphi_t(x), \varphi_t(y)\big)\big)_{t>0} \text{ bounded}\}.$$

We denote by $\overline{\mathscr{F}}$ (respectively $\widetilde{\mathscr{F}}$) the space of stable leaves on $\overline{M}$ (respectively $\widetilde{M}$).

In the triple of points model of $\overline{M}$, given $p = (x_-, x_t, x_+) \in \overline{M}$, one easily checks that

$$\overline{g}(x) = \{(y_-, y_t, y_+) \in \overline{M},\ y_- = x_-,\ y_+ = x_+\},$$
$$\overline{f}(p) = \{(y_-, y_t, y_+) \in \overline{M},\ y_+ = x_+\}.$$

It follows that $\overline{\mathscr{G}}$ identifies with the space of pair of distinct points in $\partial\mathbb{H}^2$ and $\overline{\mathscr{F}}$ identifies with $\partial\mathbb{H}^2$.

In the $\mathrm{PSL}(2,\mathbb{R})$ model of $\overline{M}$, a geodesic leaf is conjugate to the Cartan subgroup

$$A = \left\{ \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix},\ t \in \mathbb{R} \right\}.$$

A stable leaf is conjugated to the parabolic subgroup

$$P = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix},\ a > 0,\ b \in \mathbb{R} \right\}.$$

Remark 1. One can define these foliations topologically. To do so, one has to use the boundary $\partial_\infty\Gamma$ of Γ. It gives an identification between $\overline{M}$ and $\partial_\infty\Gamma^{(3)}$.

We are now able to define the pcf projective structures:

Definition 2.
- A projective structure on M is foliated if the developing map sends each geodesic leaf $\widetilde{g} \in \widetilde{\mathscr{G}}$ into a projective line and each stable leaf $\widetilde{f} \in \widetilde{\mathscr{F}}$ into a projective plane.
- A foliated projective structure on M is properly convex (pcf) if the image by the developing map of each stable leaf is a convex proper.
- Two pcf projective structures on M are equivalent if there exists a projective homeomorphism $h : M \longrightarrow M$ isotopic to the identity which respect the foliations.

Finally, we denote by $\mathscr{P}_{pcf}(M)$ the set of equivalence class of pcf projective structures on M.

1.2 Examples of projective structures on M

One can easily construct projective structures on M whose holonomy factor through a Fuchsian representation. To do so, let $\iota : \Gamma \longrightarrow \mathrm{PSL}_2(\mathbb{R})$ be a Fuchsian representation, $\rho : \mathrm{PSL}_2(\mathbb{R}) \longrightarrow \mathrm{PSL}(4,\mathbb{R})$ be a morphism and $x \in \mathbb{RP}^3$ a point such that

$$\mathrm{Stab}_{\mathrm{PSL}_2} := \{g \in \mathrm{PSL}_2(\mathbb{R}),\ \rho(g)x = x\} \text{ is finite.}$$

Then the map

$$\begin{aligned} \mathrm{dev} : \overline{M} = \mathrm{PSL}_2(\mathbb{R}) &\longrightarrow \mathbb{RP}^3 \\ g &\longmapsto \rho(g)x \end{aligned}$$

defines a projective structure on M whose holonomy factors through $\rho \circ \iota$.

Moreover, the developing map sends geodesic leaves (respectively stable leaves) into projective lines (respectively projective planes) if and only if $\mathrm{dev}(A)$ (respectively $\mathrm{dev}(P)$) is contained in a projective line (resp. projective planes).

1.2.1 Diagonal example

Consider

$$\begin{aligned} \rho : \mathrm{PSL}_2(\mathbb{R}) &\longrightarrow \mathrm{PSL}(4,\mathbb{R}) \\ g &\longmapsto \mathrm{diag}(g,g) \end{aligned}$$

and set $x := [1:0:0:1] \in \mathbb{RP}^3$. One easily checks that

- $\mathrm{Stab}_{\mathrm{PSL}_2} = \{\mathrm{Id}\}$.
- $\mathrm{dev}(A) = \big\{[u:0:0:1],\ u>0\big\} \subset \mathbb{RP}^1$.
- $\mathrm{dev}(P) = \big\{[1:0:u:v],\ u \in \mathbb{R}, v>0\big\} \subset \mathbb{RP}^2$.

However, the set $\mathrm{dev}(P)$ is not proper, so the associated projective structure fails to be properly convex.

1.2.2 Non-convex irreducible example

Identify $\mathbb{R}^4$ with the space of degree 3 homogeneous polynomials in 2 variables. It provides a canonical action of $\mathrm{PSL}_2(\mathbb{R})$ on $\mathbb{R}^4$ by pre-composing a polynomial with an element of $\mathrm{PSL}_2(\mathbb{R})$. This action defines the irreducible representation $\rho_4 : \mathrm{SL}_2(\mathbb{R}) \longrightarrow \mathrm{PSL}(4,\mathbb{R})$

Taking $x = [Q] = [XY(X+Y)]$ (so Q has three distinct real roots), one gets a projective structure whose holonomy lies in the Hitchin component but which is not convex.

1.2.3 Convex irreducible example

Consider the same ρ_4 as previously but set $x = [R] = [X(X^2+Y^2)] = [1:0:1:0] \in \mathbb{RP}^3$. One obtains:

- $\mathrm{dev}(A) = \big\{[e^{3t}:0:e^{-t}:0],\ t \in \mathbb{R}\big\} = \big\{[0:1:0:u],\ u>0\big\} \subset \mathbb{RP}^1$.
- $\mathrm{dev}(P) = \big\{[a^3+b^2a : 2b : a^{-1} : 0],\ t \in \mathbb{R}\big\}$ which is the projectivization of the convex cone $\{(\alpha,\beta,\gamma,0) \in \mathbb{R}^4,\ \beta^2 - 4\alpha\beta < 0\}$.

It follows that this defines a properly convex projective structure on M.

In order to generalize this construction to all Hitchin representations, one wants to describe this example as a map

$$\mathrm{dev} : \partial_\infty \Gamma^{(3)} \longrightarrow \mathbb{RP}^3.$$

To do so, identify $\mathbb{R}^2$ with the set of degree 1 homogeneous polynomials in 2 variables and consider the Veronese embedding

$$\begin{array}{rcl} \xi_1 : \mathbb{RP}^1 & \longrightarrow & \mathbb{RP}^3 \\ [S] & \longmapsto & [S^3] \end{array}$$

which is ρ_4-equivariant (so in particular $(\rho_4 \circ \iota)$-equivariant). The Veronese embedding lifts to a flag curve

$$\xi = (\xi_1, \xi_2, \xi_3) : \mathbb{RP}^1 \longrightarrow \mathrm{Flag}(\mathbb{R}^4)$$

where $\mathrm{Flag}(\mathbb{R}^4)$ is the space of complete flags in $\mathbb{R}^4$ and

- $\xi_1(S)$ is the line of polynomials divisible by S^3.
- $\xi_2(S)$ is the plane of polynomials divisible by S^2.
- $\xi_3(S)$ is the hyperplane of polynomials divisible by S.

Moreover, the orbits of $\mathrm{PSL}_2(\mathbb{R})$ in $\mathbb{RP}^3$ are:

- One open orbit consisting of polynomials with three distinct real roots (hence the set of points in $\mathbb{RP}^3$ contained in exactly 3 pairwise distinct $\xi_3(t)$).
- One open orbit consisting of polynomials with one real root and two complex conjugates (hence the set of points in $\mathbb{RP}^3$ contained in exactly one $\xi_3(t)$ and no $\xi_2(t)$).
- One relatively closed orbit $\bigcup_{t \in \partial_\infty \Gamma} \xi_2(t) \setminus \xi_1(t)$.
- The closed orbit $\xi_1\left(\partial_\infty \Gamma\right)$.

Note that the two open orbits are connected components of the complementary of the discriminant (ruled) surface $\bigcup_{t \in \partial_\infty \Gamma} \xi_2(t)$.

With this picture, the irreducible non-convex example is easy to recover. In fact, it is given by

$$\begin{array}{rcl} \mathrm{dev'} : \partial_\infty \Gamma^{(3)} & \longrightarrow & \mathbb{RP}^3 \\ (x_-, x_t, x_+) & \longmapsto & \xi_3(x_-) \cap \xi_3(x_t) \cap \xi_3(x_+) \end{array}$$

For the convex irreducible example, the situation is more complicated.

1.2.4 Convex irreducible example revisited

Fix $(x_-,x_t,x_+) \in \partial_\infty\Gamma^{(3)}$. Using the identification $\partial_\infty\Gamma \cong \mathbb{RP}^1$, one can see the flag curve ξ as a map from $\partial_\infty\Gamma$ to Flag$(\mathbb{R}^4)$.

Moreover, the image by ξ of distinct points giving transverse flags, for each $t \neq t' \in \partial_\infty\Gamma$, $\xi_3(t)\cap\xi_2(t')$ gives a single point in $\mathbb{RP}^3$. **Here, we identify linear subspaces of $\mathbb{R}^4$ with their image in $\mathbb{RP}^3$.**

Consider the curve

$$\begin{aligned}\mathscr{D} &:= \{\xi_2(t)\cap\xi_3(x_+),\ t\in\partial_\infty\Gamma,\ t\neq x_+\}\\ &= \Big(\bigcup_{\substack{t\in\partial_\infty\Gamma,\\ t\neq x_+}} \xi_2(t)\Big)\cap\xi_3(x_+)\subset\mathbb{RP}^3\end{aligned}$$

Adding the point $\xi_1(x_+)\in\mathbb{RP}^3$ to $\mathscr{D}$, one gets a curve which bound a strictly convex proper set in $(\xi_3(t))\cong\mathbb{RP}^2$.

The tangent line to $\mathscr{D}$ passing through $\xi_2(x_-)\cap\xi_3(x_+)\in\mathscr{D}$ intersects $\xi_2(x_+)$ in a point $p=\xi_3(x_-)\cap\xi_2(x_+)\in\mathbb{RP}^3$.

Finally we define dev(x_-,x_t,x_+) as the intersection of the projective line L_1 passing through p and $\xi_2(x_t)\cap\xi_3(x_+)\in\mathscr{D}$ and the projective line L_2 passing through $\xi_2(x_-)\cap\xi_3(x_+)\in\mathscr{D}$ and $\xi_1(x_+)$ (see Figure 1.2). More explicitly, we have

$$\mathrm{dev}(x_-,x_t,x_+) = \big((\xi_2(x_-)\cap\xi_3(x_+))\oplus\xi_1(x_+)\big)\cap\big((\xi_3(x_-)\cap\xi_2(x_+))\oplus(\xi_2(x_t)\cap\xi_3(x_+))\big).$$

Note that with this description, one easily checks that the geodesic leaf passing through (x_-,x_t,x_+) is sent into L_2 and the stable leaf passing through (x_-,x_t,x_+) is sent into the properly convex set bounded by $\mathscr{D}$.

1.3 From $\mathscr{H}(\Sigma,\mathbf{PSL}(4,\mathbb{R}))$ to $\mathscr{P}_{pcf}(M)$

Extending the previous construction to the whole Hitchin component is now very easy. In fact, it follows from a result of Labourie [**?**] that given a Hitchin representation $\rho\in\mathscr{H}(\Sigma,\mathrm{PSL}(4,\mathbb{R}))$, there exists a unique hyperconvex curve

$$\xi_1 : \partial_\infty\Gamma \longrightarrow \mathbb{RP}^3$$

which lifts to a flag curve

$$\xi : \partial_\infty\Gamma \longrightarrow \mathrm{Flag}(\mathbb{R}^4).$$

Moreover, this curve satisfies the transverse condition (namely that each pair of distinct points $t,t'\in\partial_\infty\Gamma$ are sent into transverse flags). So one can define the following developing map:

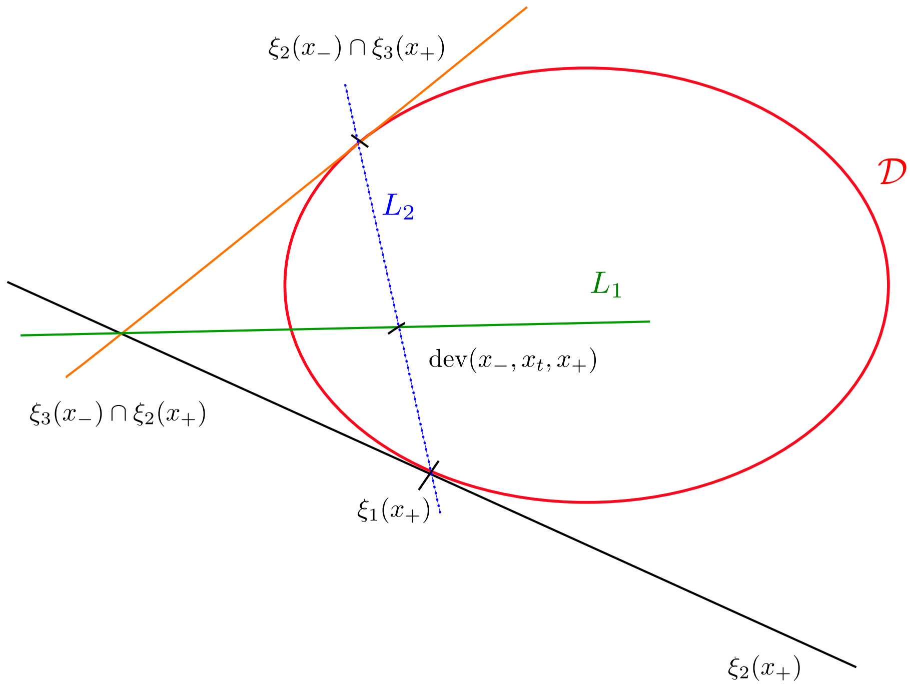

Fig. 1.2 The developing map.

$$\mathrm{dev}(x_-,x_t,x_+) = \big((\xi_2(x_-)\cap\xi_3(x_+))\oplus\xi_1(x_+)\big)\cap\big((\xi_3(x_-)\cap\xi_2(x_+))\oplus(\xi_2(x_t)\cap\xi_3(x_+))\big).$$

This map is ρ-equivariant because the flag curve is. Moreover it is clear that it is foliated (for the same arguments as previously).

So one gets a map

$$\mathscr{H}(\Sigma, \mathrm{PSL}(4,\mathbb{R})) \longrightarrow \mathscr{P}_{pcf}(M).$$

It is proved in [**?**] that this map is one-to-one. It means in particular that the holonomy of a pcf projective structure on M factors through a Hitchin representation.

1.4 The PSp$(4,\mathbb{R})$ case

Let $(\mathbb{R}^4,\omega)$ be the 4-dimensional real space endowed with a symplectic form ω. The subgroup of PSL$(4,\mathbb{R})$ preserving ω is the symplectic group PSp$(4,\mathbb{R})$.

Given a line $L \subset \mathbb{R}^4$ (necessarily isotropic), its orthogonal complement with respect to ω is a 3-dimensional subspce $L^\perp \subset \mathbb{R}^4$. It implies that each point $x \in \mathbb{RP}^3$ defines a projective plane $x^\perp \cong \mathbb{RP}^2$ containing x.

By considering the tangent space of $x^\perp$ insides the tangent space of $\mathbb{RP}^3$ at x, one can define a hyperplane distribution in $T\mathbb{RP}^3$. This distribution corresponds to a contact structure on $\mathbb{RP}^3$ that we call *contact projective structure*.

Given a representation $\rho \in \mathscr{H}(\Sigma, \mathrm{PSp}(4,\mathbb{R}))$, the associated flag curve ξ takes value in the complete flag manifold associated to $\mathrm{PSp}(4,\mathbb{R})$. More precisely, ξ associates to each point of $\partial_\infty \Gamma$ a complete $\mathrm{PSp}(4,\mathbb{R})$-flag $(l, \mathscr{L}, l^\perp)$ where $l \subset \mathbb{R}^4$ is a line, $\mathscr{L}$ is a Lagrangian plane and $l^\perp$ is the orthogonal of l.

The same construction as previously gives rise to a pcf contact projective structure on M where the geodesic leaves are sent into (projection of) Lagrangian planes.