

Maximal Representations

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Abstract Following the work of Burger, Iozzi and Wienhard [BIW10], and Burger, Iozzi, Labourie and Wienhard [BILW05] we use bounded cohomology to define maximal representations. This allows us to list the possible Zariski closure of a maximal representation in $\mathrm{Sp}(4, \mathbb{R})$. We deduce from the bounded cohomological framework the existence of reasonably well behaved boundary maps, and discuss how boundary maps can be used to prove topological and geometric statements about maximal representations. The note contains no new result.

1 Introduction

Maximal representations are interesting components of the representation variety $\mathrm{Hom}(\Gamma, G)$ where Γ is the fundamental group $\Gamma = \pi_1(\Sigma)$ of an hyperbolic surface Σ , which might or might not have boundary, and G is an Hermitian Lie Group. We fix once and for all an auxiliary finite volume hyperbolization of Σ and a realization of Γ as a discrete subgroup of $\mathrm{PSL}_2(\mathbb{R}) = \mathrm{Isom}^+(\mathbf{H}^2)$, this is not really necessary, but helps streamlining many statements.

In this note we will mostly discuss maximal representations in $\mathrm{PSL}_2(\mathbb{R})$ and $\mathrm{Sp}(4, \mathbb{R})$. We begin discussing some of the ideas involved in the definition of maximal representations in the case of $\mathrm{PSL}_2(\mathbb{R})$. We will see that they correspond to points in the Teichmüller space. We will be naturally lead to the introduction of Hermitian Lie groups, a class containing $\mathrm{Sp}(4, \mathbb{R})$, and, in Section 3, we will discuss some features of the symmetric space of $\mathrm{Sp}(4, \mathbb{R})$ that generalize to all Hermitian Lie groups and play an important role in the study of maximal representations. Bounded cohomology will first appear in Section 4 and will be immediately used, in Section 5, to determine the possible Zariski closure of a maximal representation

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with values in $\mathrm{Sp}(4, \mathbb{R})$. In Section 6 we will discuss how the bounded cohomological information allows to deduce properties of equivariant boundary maps. We finish exemplifying how to obtain geometric and topological information on the representations from the existence of a well behaved boundary map in Section 7.

For the sake of concreteness, most of the statements are only given in the case $\mathrm{Sp}(4, \mathbb{R})$, despite they can be easily generalized. We won't give any complete proof, but we hope to point out the major steps and ingredients and to convey at least some of the nice ideas in the area. We refer the reader to [BIW15, Har13] for more detailed introductions to the subject.

2 A motivating example

Probably the most familiar instance of maximal representations is the Teichmüller space, as we will now shortly discuss.

We denote by \mathbf{H}^2 the (upperhalf plane model of the) hyperbolic plane, and we focus on the group $G = \mathrm{PSp}_2(\mathbb{R}) = \mathrm{PSL}_2(\mathbb{R}) = \mathrm{Isom}^+(\mathbf{H}^2)$. Let Γ_g denote the fundamental group of a compact surface of genus g . It is by now well understood that the algebraic variety

$$\mathrm{Hom}(\Gamma_g, \mathrm{PSL}_2(\mathbb{R})) = \left\{ (x_1, \dots, x_{2g}) \in \mathrm{PSL}_2(\mathbb{R})^{2g} \mid \prod_{i=0}^{g-1} [x_{2i}, x_{2i+1}] = 1 \right\} \quad (1)$$

parametrizing the space of homomorphisms from the discrete group Γ_g to $\mathrm{PSL}_2(\mathbb{R})$ has $4g - 3$ connected components, which are distinguished by the Euler class. Bill Goldman, in his thesis, understood the geometric significance of representations with maximal Euler class.

Theorem 1 ([Gol80]). *A representation $\rho : \Gamma \rightarrow \mathrm{PSL}_2(\mathbb{R})$ is the holonomy of an hyperbolic structure if and only if $e(\rho) = 2g - 2$.*

What will be important for us is that the Euler number is a characteristic invariant that selects two particularly interesting components of the representation variety.

It is possible to reinterpret the Euler number of a representation ρ as its *volume* in the following sense (cfr. [BIW15, Section 3.7]). Let $\omega \in \Omega^2(\mathbf{H}^2, \mathbb{R})^{\mathrm{PSL}_2(\mathbb{R})}$ be the Riemannian volume form of the hyperbolic plane \mathbf{H} . For any smooth ρ -equivariant map $f : \tilde{\Sigma} \rightarrow \mathbf{H}^2$ we can define the volume of the representation ρ as

$$T(\rho) = \mathrm{vol}(\rho) = \frac{1}{\pi} \int_{\Sigma} f^* \omega. \quad (2)$$

Using uniqueness of geodesics in \mathbf{H}^2 one can construct ρ -equivariant homotopies between any two ρ -equivariant maps and thus show that the invariant $T(\rho)$ doesn't depend on the map f but only on the representation ρ . The invariant $T(\rho)$ is referred to as the *Toledo invariant*.

Through the next (commutative) diagram we can find a more cohomological reinterpretation of the volume of the representation ρ . Recall that the cohomology of a group G is the homology of the complex of G -invariant functions $C^*(G, \mathbb{R}) = \{f : G^{*+1} \rightarrow \mathbb{R}\}^G$, and, in case of countable groups, is isomorphic to the singular cohomology of a $K(G, 1)$: for example in the case of the fundamental group Γ_g of a compact surface Σ , the cohomology $H^*(\Gamma_g, \mathbb{R})$ is isomorphic to the singular cohomology of Σ . Each differential form on a symmetric space \mathcal{X} induces a cohomology class on $\text{Isom}(\mathcal{X})$ defined choosing a basepoint and integrating on geodesic simplices [Dup76].

$$\begin{array}{ccc}
 H_c^2(G, \mathbb{R}) & \xleftarrow{[\text{Dup76}]} & \Omega^2(\mathbf{H}^2, \mathbb{R})^{\text{PSL}_2(\mathbb{R})} & (3) \\
 \downarrow \rho^* & & \downarrow f^* & \\
 H^2(\Gamma, \mathbb{R}) & & \Omega^2(\tilde{\Sigma}/\Gamma, \mathbb{R}) & \\
 \downarrow & & \downarrow & \\
 H_{\text{sing}}^2(\Sigma, \mathbb{R}) & \xrightarrow{\cong} & H_{\text{dR}}^2(\Sigma, \mathbb{R}) & \\
 & \searrow \langle [\Sigma], \cdot \rangle & \downarrow J_\Sigma & \\
 & & \mathbb{R} &
 \end{array}$$

The power of group cohomology is that it is relatively easy to prove that most natural diagrams, as diagram (3) in this case, commute. This is the idea behind the fundamental theorem of homological algebra. We won't have time to discuss it here in detail, apart from pointing out places where it plays a crucial role, and refer to [Bro82, Gui80, Mon01] for a thorough discussion in different settings. It is however worth mentioning that most cohomological theories are set up so that an analogue of the fundamental theorem of homological algebra works.

We just introduced a differential geometric, and a cohomological way of selecting components of the representation variety. There are two directions in which this ideas can be generalized. On the one hand, we can also approach the study of representations of fundamental groups of surfaces with boundary. On the other we can consider different Lie groups G as targets. The only requirement is that the associated symmetric space admits a G -invariant differential two form that can be used to define the volume of a representation. Such Lie groups are precisely the *Hermitian Lie groups* whose basic properties will be discussed in the next section. We will discuss how bounded cohomology can help integrating differential forms on non-compact surfaces in Section 4.

3 Hermitian Lie groups

Definition 1. A noncompact simple Lie group G is *Hermitian* if one of the following equivalent properties holds:

1. The symmetric space G/K admits a G -invariant complex structure;
2. the symmetric space G/K is a Kähler manifold;
3. there is a G -invariant differential two form on G/K .

Hermitian Lie groups are classified. In addition to two exceptional Lie groups $E_6(-14)$ and $E_7(-25)$, there are, up to isogeny, four classical families: $SU(p, q)$, $SO(2, n)$, $Sp(2n, \mathbb{R})$, $SO^*(2n)$.

We will now describe in a bit more detail the symmetric space associated to $Sp(4, \mathbb{R}) = SO^0(2, 3)$ highlighting some general features that are useful in the study of maximal representations. We refer the reader to [Kor00] for the general case. We identify $Sp(4, \mathbb{R})$ with the subgroup of $GL(4, \mathbb{R})$ consisting of matrices preserving the symplectic form $\langle \cdot, \cdot \rangle$ represented, with respect to the standard basis, by the matrix $\begin{pmatrix} 0 & \text{Id}_2 \\ -\text{Id}_2 & 0 \end{pmatrix}$. The elements of $Sp(4, \mathbb{R})$ have expression

$$\begin{pmatrix} A & B \\ \mathbb{C} & D \end{pmatrix} \quad \begin{array}{l} {}^tAD - {}^tCB = \text{Id} \\ {}^tAC = \mathbb{C}^tA \\ {}^tBD = {}^tDB. \end{array} \quad (4)$$

We extend by linearity the symplectic form $\langle \cdot, \cdot \rangle$ to a symplectic form $\langle \cdot, \cdot \rangle_{\mathbb{C}}$ on \mathbb{C}^4 and denote by $\mathcal{L}(\mathbb{C}^4)$ (resp. $\mathcal{L}(\mathbb{R}^4)$) the complex (resp. real) Lagrangians, namely the maximal isotropic subspaces for $\langle \cdot, \cdot \rangle_{\mathbb{C}}$:

$$\mathcal{L}(\mathbb{C}^4) = \{V \in \text{Gr}_2(\mathbb{C}^4) \mid \langle \cdot, \cdot \rangle|_{V \times V} = 0\}. \quad (5)$$

The group $Sp(4, \mathbb{R})$ acts on $\mathcal{L}(\mathbb{C}^4)$, and the real Lagrangians $\mathcal{L}(\mathbb{R}^4)$ naturally sit as an half-dimensional submanifold of $\mathcal{L}(\mathbb{C}^4)$ preserved by the $Sp(4, \mathbb{R})$ action. The symplectic form $\langle \cdot, \cdot \rangle_{\mathbb{C}}$ induces an Hermitian form

$$h(v, w) = i \langle v, \bar{w} \rangle_{\mathbb{C}}$$

which is also preserved by the action of $Sp(4, \mathbb{R})$. A model for the symmetric space associated to $Sp(4, \mathbb{R})$ can be given as the open, semialgebraic subset of $\mathcal{L}(\mathbb{C}^4)$ given by

$$\mathbb{X} = \{V \in \mathcal{L}(\mathbb{C}^4) \mid h|_V \text{ is positive definite}\}. \quad (6)$$

Notice that $\mathcal{L}(\mathbb{R}^4)$ is contained in the topological closure $\partial\mathbb{X}$ of \mathbb{X} in $\mathcal{L}(\mathbb{C}^4)$ and is the only closed $Sp(4, \mathbb{R})$ -orbit in $\partial\mathbb{X}$. Indeed the real Lagrangians are the *Shilov boundary* of the symmetric space \mathbb{X} , and can also be characterized by the property that it is the unique minimal subset of the closure $\bar{\mathbb{X}}$ with the property that all continuous functions on $\bar{\mathbb{X}}$ that are holomorphic on \mathbb{X} satisfy $|f(x)| \leq \max_{y \in \mathcal{L}(\mathbb{R}^4)} f(y)$

for all $x \in \mathbb{X}$. What we just described in our specific case is the *Borel embedding* of the symmetric space, and has an analogue for each Hermitian symmetric space.

A more concrete model is the *Siegel upperhalf plane*:

$$\mathcal{X} = \{X + iY \mid X \in \text{Sym}_2(\mathbb{R}), Y \in \text{Sym}_2(\mathbb{R}) \text{ positive definite}\}. \quad (7)$$

The group $\text{Sp}(4, \mathbb{R})$ acts on \mathcal{X} by fractional linear transformations and the identification of \mathbb{X} with \mathcal{X} is given by restricting the affine patch $\text{Sym}_2(\mathbb{C})$ of $\mathcal{L}(\mathbb{C}^4)$ parametrizing subspaces transverse to $\text{span}\{e_1, e_2\}$: each such Lagrangian is uniquely spanned by the columns of the matrix $\begin{pmatrix} Z \\ \text{Id} \end{pmatrix}$ for some complex symmetric matrix Z . It is easy to verify that the restriction of h to the subspace associated to the symmetric matrix Z is positive definite if and only if the imaginary part of Z is positive definite.

Clearly \mathcal{X} admits a $\text{Sp}(4, \mathbb{R})$ -invariant complex structure, and hence is a Hermitian symmetric space. The Kähler form ω at the point $Z = X + iY \in \mathcal{X}$ has expression

$$\omega = \text{tr}(Y^{-1}dZ \wedge Y^{-1}d\bar{Z}) \quad (8)$$

where the multiplication is intended as matrix multiplication (cfr. [Sie43, Section 11]). This is normalized so that the minimal holomorphic sectional curvature is -1.

As in Section 2, if Σ is compact, we can use the Kähler form ω to define a volume for a representation $\rho : \Gamma_g \rightarrow \text{Sp}(4, \mathbb{R})$. We chose a ρ -equivariant map $f : \tilde{\Sigma} \rightarrow \mathcal{X}$ and define

$$T(\rho) = \frac{1}{\pi} \int_{\Sigma} f^* \omega \quad (9)$$

This is known as the *Toledo invariant*: in [Tol89] Toledo introduced this definition of volume of a representation and combined it with ideas of Gromov and Thurston on bounded cohomology and simplicial volume to prove that what we would now call maximal representations in $\text{PU}(1, n) = \text{Isom}(\mathbf{H}_{\mathbb{C}}^n)$ stabilize complex geodesics.

We will see in the next section that for any representation $\rho : \Gamma \rightarrow \text{Sp}(4, \mathbb{R})$ we have $|T(\rho)| \leq 4g - 4$. *Maximal representations* are precisely those homomorphisms for which equality is attained.

3.1 Some totally geodesic subspaces

We finish this section singling out some totally geodesic subspaces of \mathcal{X} that will play a crucial role in the following. These are all *tight* subspaces as defined in [BIW09].

3.1.1 Polydiscs and diagonal discs

An important feature of Hermitian symmetric spaces is that maximal flats complexify to *polydiscs*: in our setting this means that each two dimensional totally geodesic flat subspace $\mathbb{R}^2 \rightarrow \mathcal{X}$ is contained in a unique totally geodesic holomorphically embedded copy of $\mathbf{H}^2 \times \mathbf{H}^2$. If \mathbf{H} denotes the upperhalf plane model of the hyperbolic space, an example of such a polydisc is given by

$$\mathbf{H}^2 \times \mathbf{H}^2 \rightarrow \mathcal{X} \quad (\lambda, \mu) \mapsto \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}. \quad (10)$$

We will refer to this polydisc as a model polydisc since each other polydisc is translate of our model by an element in $\mathrm{Sp}(4, \mathbb{R})$.

It is easy to verify that the Kähler form ω restricts to the product of the Kähler forms of the two factors. This implies that if we consider the *diagonal disc*

$$\mathbf{H}^2 \rightarrow \mathcal{X} \quad \lambda \mapsto \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}. \quad (11)$$

the restriction of the Kähler form of \mathcal{X} to \mathbf{H}^2 , is twice the Kähler form of \mathbf{H}^2 . Using this information it is easy to verify that if we denote by $\Delta : \mathrm{SL}(2, \mathbb{R}) \rightarrow \mathrm{Sp}(4, \mathbb{R})$ the diagonal inclusion (the unique homomorphism equivariant with the diagonal disc), we get that for any hyperbolization $\rho : \Gamma \rightarrow \mathrm{SL}_2(\mathbb{R})$, the composition $\Delta \circ \rho : \Gamma \rightarrow \mathrm{Sp}(4, \mathbb{R})$ is a maximal representation.

3.1.2 The irreducible representation

Another example of totally geodesic subspace of \mathcal{X} isomorphic to the Poincaré disc is the subspace associated to the image of the irreducible representation $\iota : \mathrm{SL}(2, \mathbb{R}) \rightarrow \mathrm{Sp}(4, \mathbb{R})$: as opposed to the image of a diagonal disc this totally geodesic copy of \mathbf{H}^2 is not holomorphically embedded, however it also has the property that the Kähler form restricts to twice the Kähler form of \mathbf{H}^2 and hence the composition of any hyperbolization with the homomorphism ι is a maximal representation.

4 A crash course in bounded cohomology

Let G be a locally compact, second countable group (in this note G will either be $\Gamma = \pi_1(\Sigma)$ endowed with its discrete topology or $\mathrm{Sp}(4, \mathbb{R})$). The continuous bounded cohomology of G with real coefficients is the cohomology of the complex

$$C_{c,b}^*(G, \mathbb{R})^G = \{f : G^{*+1} \rightarrow \mathbb{R} \mid \text{continuous, bounded, } G\text{-invariant}\}. \quad (12)$$

The continuous bounded cohomology of G is endowed with a canonical seminorm $\|\cdot\|_\infty$ defined by the infimum of the l^∞ -norms of the representatives. Burger and Monod proved that this seminorm is a norm in degree 2 [BM02], however, at least for Γ , this is not a norm in degree 3 [Som97], namely there exist classes of zero norm. In general the (continuous) bounded cohomology is a quite mysterious theory and many basic questions are still open, however, in degree two, the picture for Lie groups is completely understood:

Theorem 2 ([BM02, vE53]). *Let G be a semisimple Lie group without compact factors, then*

$$H_{c,b}^2(G, \mathbb{R}) = H_c^2(G, \mathbb{R}) = \Omega^2(G/K, \mathbb{R})^G \quad (13)$$

In particular $H_{c,b}^2(\mathrm{Sp}(4, \mathbb{R}), \mathbb{R}) \cong \mathbb{R}$. However it is worth keeping in mind that the bounded cohomology of discrete groups is rather wild: $H_b^2(\Gamma, \mathbb{R})$ is infinite dimensional both in the case of surface groups and in the case of free groups [Som97].

4.1 Toledo invariant and Milnor-Wood type inequalities, compact surfaces

The bounded cohomological class corresponding to the Kähler form is called the *bounded Kähler class* and usually denoted by κ_G^b . In the case of classical Lie groups the norm of the bounded Kähler class was computed by Domic and Toledo:

Theorem 3 ([DT87]). *Let G be a semisimple Lie group all whose factors are of Hermitian type. Then*

$$\|\kappa_G^b\|_\infty = \mathrm{rank}(G). \quad (14)$$

Since bounded cochains form a subcomplex of ordinary cochains, there is a natural map, the *comparison map*, between bounded cohomology and ordinary cohomology:

$$c : H_b^2(\Gamma_g, \mathbb{R}) \rightarrow H^2(\Gamma_g, \mathbb{R}). \quad (15)$$

We already pointed out in Section 2 that the cohomology of the group Γ_g is isomorphic to the cohomology of the surface Σ . In particular we can follow the arrows

$$H_{\mathrm{cb}}^2(\mathrm{Sp}(4, \mathbb{R}), \mathbb{R}) \xrightarrow{\rho^*} H_b^2(\Gamma_g, \mathbb{R}) \xrightarrow{c} H^2(\Gamma_g, \mathbb{R}) \xrightarrow{\cong} H^2(\Sigma, \mathbb{R}) \quad (16)$$

and use the pairing of the singular cohomology with the singular homology to define the Toledo invariant as

$$T(\rho) = \langle c \circ \rho^*(\kappa_{\mathrm{Sp}(4, \mathbb{R})}^b), [\Sigma] \rangle. \quad (17)$$

Bounded cohomology turns out to be a precious tool for the first time now: it allows to easily prove a *Milnor-Wood* type inequality for the Toledo number (this is in general hard to achieve with differential geometric tools only): we know that

the norm of $\kappa_{\mathrm{Sp}(4,\mathbb{R})}^b$ is 2, moreover one immediately checks from the very definition that ρ^* is norm non-increasing. In particular we get

$$|T(\rho)| \leq \|\rho^*(\kappa_{\mathrm{Sp}(4,\mathbb{R})}^b)\|_\infty \|\Sigma\|_1. \quad (18)$$

Since the simplicial volume of Σ is $2g - 2$ this reads $|T(\rho)| \leq 4g - 4$.

4.2 Surfaces with boundary

The most important reason why bounded cohomology is needed in the definition of maximal representations will be clear in Section 6: it allows to deduce informations on boundary maps. Another advantage of bounded cohomology with respect to ordinary cohomology is that it allows to treat compact and non-compact surfaces within the same framework.

If the surface Σ is non compact, $H^2(\Sigma, \mathbb{R}) = H^2(\mathbb{F}_2, \mathbb{R})$ is trivial, hence the cohomological definition of maximality discussed in Section 2 doesn't apply. Also the differential geometric definition of the Toledo invariant gives problems: the value of the integral isn't anymore independent on the choice of the equivariant map (as an example one can consider f being the lift to the universal cover of a retraction on a spine of the hyperbolic surface). However amenable groups (for example the peripheral subgroups of the fundamental group of the surface) are invisible to bounded cohomology, and this allows to prove the following:

Theorem 4 ([BBI⁺14, KK15]). *Let Σ be a surface of finite type with boundary $\partial\Sigma$. There is an isometric isomorphism*

$$i^* : H_b^2(\Sigma, \mathbb{R}) \rightarrow H_b^2(\Sigma, \partial\Sigma, \mathbb{R}) \quad (19)$$

Since $H_2(\Sigma, \partial\Sigma)$ is one dimensional generated by the relative fundamental class $[\Sigma, \partial\Sigma]$, one can define the Toledo invariant as

$$T(\rho) = \langle c \circ i^* \circ \rho^*(\kappa_{\mathrm{Sp}(4,\mathbb{R})}^b), [\Sigma, \partial\Sigma] \rangle \quad (20)$$

and prove, also in this case, that the Toledo invariant satisfies a Milnor Wood type inequality. See [BIW10, Section 3] for more detail.

5 Zariski closure

The same arguments that allow to prove the Milnor-Wood inequality also allow to understand the possible Zariski closures of a maximal representation. Indeed the pullback map ρ^* factorst through the restriction to the Zariski closure of the representation ρ . Denoting by $\overline{\rho(\Gamma)}^Z$ the Zariski closure of the image of a representation

$\rho : \Gamma \rightarrow G$ and by $i : \overline{\rho(\Gamma)}^Z \rightarrow G$ the inclusion, we have that ρ^* can be written as the composition

$$H_{\text{cb}}^2(\text{Sp}(4, \mathbb{R})) \xrightarrow{i^*} H_{\text{cb}}^2(\overline{\rho(\Gamma)}^Z) \xrightarrow{\rho^*|} H_{\text{b}}^2(\Gamma_{\text{g}}, \mathbb{R}) \quad (21)$$

This observation, together with Theorem 2 implies that, if ρ is maximal and the Zariski closure $\overline{\rho(\Gamma)}^Z$ is simple, then it needs to be a group of Hermitian type: otherwise we would get that $H^2(\overline{\rho(\Gamma)}^Z, \mathbb{R}) = 0$ and the same would necessarily be true for the pullback $\rho^*(\kappa_{\text{Sp}(4, \mathbb{R})}^b)$.

It follows from the proof of the Milnor-Wood inequality that we discussed in Section 4 that if ρ is maximal then $\|\rho^*(\kappa_{\text{Sp}(4)}^b)\|_{\infty} = \|\kappa_{\text{Sp}(4)}^b\|_{\infty}$. We observed that the ρ^* factors through the inclusion i^* of the Zariski closure of the image of ρ . Since both i^* and ρ^* are norm non increasing, we deduce that, whenever ρ is maximal, the inclusion $i : \overline{\rho(\Gamma)}^Z \rightarrow \text{Sp}(4, \mathbb{R})$ needs to be a *tight* homomorphism: the induced map in continuous bounded cohomology is isometric [BIW09].

Tight homomorphism were defined and studied by Burger, Iozzi and Wienhard in [BIW09]. They also discussed the notion of tight embedding of symmetric spaces. For us these will be totally geodesic embeddings equivariant with a tight embedding of Lie groups (see [BIW09] for a more direct definition).

Determining the tightly embedded symmetric subspaces of $\text{Sp}(4, \mathbb{R})$ is easy: their rank is at most two, therefore, apart from \mathcal{X} itself, can only be \mathbf{H}^2 and $\mathbf{H}^2 \times \mathbf{H}^2$ and it turns out that the only possible tight embeddings are the ones described in Section 4. Determining the precise Zariski closure of a maximal representation is slightly more involved (since there might be factors in the compact centralizer of their image). All the details can be found in [BGPG12].

5.1 The general case

In general the symmetric spaces associated to the possible Zariski closures of a maximal representations were discussed in [HP14] building on ideas from [BIW09] and [Ham12]: in order to classify all possible tight embeddings between classical Lie groups, it is useful to consider the associated linear representation, and its decomposition in irreducible factors. It was already proven in [BIW09] that the irreducible factors of a tight representation of $\text{SL}_2(\mathbb{R})$ can only have odd weights, these up to standard Lie group inclusions, up to standard Lie group inclusions, correspond to the irreducible representations of $\text{SL}_2(\mathbb{R})$ having values in $\text{Sp}(2n, \mathbb{R})$. In his thesis [Ham12] Hamlet smartly deduced that a tight homomorphism of any other simple Lie group needs to be holomorphic. Similar ideas allowed us to prove in [HP14] that any irreducible tight representation of a semisimple Lie group essentially factors through a simple factor and finish the classification comparing the ranks of the groups involved.

6 Boundary maps

We now go back to the bounded cohomological framework, and in particular to Theorem 2 and discuss how bounded cohomology can be used to deduce information on equivariant boundary maps.

Domic and Toledo (and in wider generality Clerc and Orsted) gave an useful representative of the class in $H_{c,b}^2(G, \mathbb{R})$ corresponding to the Kähler form, which also allows to compute its norm:

Theorem 5 ([Dup76, CO03, BIW10]). *Fix any Lagrangian $l \in \mathcal{L}(\mathbb{R}^4)$. The class $\kappa_{\text{Sp}(4, \mathbb{R})}^b$ corresponding to the Kähler form is represented by the cocycle*

$$\beta_l(g_0, g_1, g_2) = \frac{1}{\pi} \int_{\Delta(g_0l, g_1l, g_2l)} \omega = \text{sign}(Q'_{g_0l, g_1l, g_2l}). \quad (22)$$

In particular $\|\kappa_{\text{Sp}(4, \mathbb{R})}^b\|_\infty = 2$.

Here $\Delta(g_0l, g_1l, g_2l)$ denotes an (ideal) geodesic triangle in \mathcal{X} whose sides are pairwise asymptotic, and point to the Lagrangian $g_i l$ in the sense that the stabilizer in $\text{Sp}(4, \mathbb{R})$ of the ray is the same as the stabilizer of the Lagrangian $g_i l$. We denote by Q'_{g_0l, g_1l, g_2l} the quadratic form on $g_0l \oplus g_1l \oplus g_2l$ given by

$$Q'_{g_0l, g_1l, g_2l}(v_0, v_1, v_2) = \langle v_0, v_1 \rangle + \langle v_1, v_2 \rangle + \langle v_2, v_0 \rangle \quad (23)$$

and by signature of a quadratic form we understand the difference of the number of positive eigenvalues and the number of negative eigenvalues. The cocycle β_l is referred to as the Kashiwara cocycle [LV80]. One can verify that if (g_0l, g_1l, g_2l) are pairwise transverse then the signature of Q'_{g_0l, g_1l, g_2l} is equal to the signature of the quadratic form Q_{g_0l, g_1l, g_2l} on g_0l given by $Q_{g_0l, g_1l, g_2l}(v) = \langle v, T_{g_0l, g_2l}^{g_1l}(v) \rangle$, where $T_{g_0l, g_2l}^{g_1l} : g_0l \rightarrow g_2l$ is the linear map defined by $v + T_{g_0l, g_2l}^{g_1l}(v) \in g_1l$.

It is possible to check that the value of the Kashiwara cocycle is a complete invariant for the action of $\text{Sp}(2n, \mathbb{R})$ on triples of pairwise transverse Lagrangians. In particular there are exactly $n + 1$ $\text{Sp}(4, \mathbb{R})$ -orbits in the set of triples of pairwise transverse Lagrangians and a triple (l_0, l_1, l_2) is *maximal* if Q_{l_0, l_1, l_2} is positive definite.

Ideas in the proof of Theorem 5. We already mentioned that Dupont [Dup76] showed that the correspondence $\Omega^2(G/K, \mathbb{R})^G \rightarrow H_c^2(G, \mathbb{R})$ can be obtained integrating on totally geodesic triangles with endpoints (g_0x, g_1x, g_2x) in the symmetric space. This was another instance of a fundamental theorem of homological algebra. Inspired by the work of Domic and Toledo [DT87], Clerc and Orsted [CO03] showed that, if the triple (g_0l, g_1l, g_2l) consists of pairwise transverse subspaces, the limit as the basepoint x converges to the Lagrangian l is well defined, and can be extended for all triples restricting to suitable tangential limits.

We will now verify the equality with the signature of a quadratic form, in the case of pairwise transverse Lagrangians. Observe that, for each element $g \in \text{Sp}(4, \mathbb{R})$, and

for each triple (l_0, l_1, l_2) of pairwise transverse Lagrangian the equality

$$\frac{1}{\pi} \int_{\Delta(l_0, l_1, l_2)} \omega = \frac{1}{\pi} \int_{\Delta(g \cdot l_0, g \cdot l_1, g \cdot l_2)} \omega \quad (24)$$

holds. A direct computation shows that, up to the symplectic action we can assume that $l_0 = \langle e_1, e_2 \rangle, l_1 = \langle e_3, e_4 \rangle$ and l_2 corresponds to $\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$. These three points belong to the boundary of the model polydisc (cfr. Section 3.1.1). The result follows combining the fact that the area of an ideal triangle in \mathbf{H}^2 is $\pm\pi$ depending on the orientation and the fact that the quadratic form Q_{l_0, l_1, l_2} is represented by the matrix $\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$. \square

Unfortunately we won't have space to discuss this in detail here, but the strenght of bounded cohomology as opposed to ordinary cohomology is that the bounded cohomology can be computed on the boundary. An exemple of this is provided in the next theorem, whose main ingredients are, again, the fundamental theorem of homological algebra and the fact that amenable groups are negligible in bounded cohomology:

Theorem 6 ([BM02]).

$$H_b^2(\pi_1(\Sigma), \mathbb{R}) = L_{\text{alt}}^\infty((\partial\mathbf{H}^2)^3, \mathbb{R})^{\pi_1(\Sigma)}. \quad (25)$$

Theorem 6 is particularly useful since it can be used to realize the pullback via measurable equivariant boundary maps:

Theorem 7 ([BI09]). *For each measurable ρ -equivariant map $\phi : \partial\mathbf{H}^2 \rightarrow \mathcal{L}(\mathbb{R}^4)$ the diagram*

$$\begin{array}{ccc} L_{\text{alt}}^\infty(\partial\mathbf{H}^2)^3, \mathbb{R})^{\pi_1(\Sigma)} & \xleftarrow{\phi^*} & \mathcal{B}(\mathcal{L}(\mathbb{R}^4)^3, \mathbb{R})^{\text{Sp}(4)} \\ \downarrow \cong & & \downarrow \\ \mathbf{mathbf{H}}_b^2(\Gamma, \mathbb{R}) & \xleftarrow{\rho^*} & \mathbf{H}_{c,b}^2(G, \mathbb{R}) \end{array} \quad (26)$$

commutes. In particular ρ is maximal if and only if for almost every positively oriented triple (x_0, x_1, x_2) we have

$$2 = \int_{\text{SL}_2(\mathbb{R})/\Gamma} \text{sign}(Q_{\phi(g \cdot x_0), \phi(g \cdot x_1), \phi(g \cdot x_2)}) dg. \quad (27)$$

By now the existence of measurable boundary maps equivariant with respect to Zariski dense representations is well established (a proof for the result needed in our setting can be found in [BI04, Proposition 7.2]), and another application of the ideas sketched in Section 5 about tight embeddings is that, when dealing with maximal representations, one can reduce to the case of Zariski dense representations.

As a consequence of Theorem 7, the information on the maximality of the representation ρ , combined with bounded cohomology, allows to deduce information on the boundary maps:

Theorem 8 ([BIW10]). *A representation $\rho : \Gamma \rightarrow \mathrm{Sp}(4, \mathbb{R})$ is maximal if and only if there exists a ρ -equivariant map $\phi : \partial\mathbf{H}^2 \rightarrow \mathcal{L}(\mathbb{R}^4)$ which is monotone, namely such that for almost every positively (resp. negatively) oriented triple (x_0, x_1, x_2) in $\partial\mathbf{H}^2$ the quadratic form $\mathcal{Q}_{\phi(x_0), \phi(x_1), \phi(x_2)}$ is positive (resp. negative) definite.*

It is possible to show that since ϕ is monotone, if Σ is compact, then ϕ is necessarily continuous and uniquely defined. If instead Σ has cusps, ϕ might not be continuous in the cusp points, but can be chosen to be right or left continuous [BIW10].

7 Application

We finish this note discussing some consequences of the existence of monotone boundary maps:

Theorem 9 ([BIW10]). *Let ρ be a maximal representation, then it is discrete and injective. If moreover Σ is compact then it is Anosov.*

Proof. We will only sketch a proof of how injectivity can be deduced from the fact that the equivariant boundary map ϕ is monotone. Let us first assume by contradiction that the representation ρ is not injective. We chose an infinite order element γ in $\ker(\rho)$ and an interval $I \subset \partial\mathbf{H}^2$ small enough so that $I, \gamma \cdot I, \gamma^2 \cdot I$ are disjoint and positively oriented. Chose x, y, z in I so that the triple (x, y, z) is positively oriented. We get that $(\gamma^2 \cdot x, \gamma \cdot y, z)$ is negatively oriented. The contradiction arises from the fact that, since γ is in the kernel of ρ , and ϕ is ρ -equivariant, $(\phi(\gamma^2 \cdot x), \phi(\gamma \cdot y), \phi(z)) = (\phi(x), \phi(y), \phi(z))$ but monotonicity would imply that the quadratic form associated to the first triple is negative definite and the one associated to the second one is positive definite.

The fact that the boundary map ϕ associated with a maximal representation is positive, in the sense that the quadratic form associated to any maximal triple is positively definite, allows to generalize the Collar Lemma valid for hyperbolicizations to all maximal representations:

Theorem 10 ([BP15, Theorem 6]). *Let $\rho : \Gamma \rightarrow \mathrm{Sp}(4, \mathbb{R})$ be maximal. For each γ, η in Γ whose axis intersect we have:*

$$(|\lambda_1(\rho(\gamma))\lambda_2(\rho(\gamma))| - 1)(|\lambda_1(\rho(\eta))\lambda_2(\rho(\eta))| - 1) \geq 1 \quad (28)$$

where for an element g in $\mathrm{Sp}(4, \mathbb{R})$ we denote by $\lambda_1(g), \lambda_2(g)$ its eigenvalues of absolute value greater than or equal to 1.

We refer the reader to [BP15] for a proof. A fundamental ingredient in this case is a matrix valued crossratio associated to each positively oriented 4-tuple (x, y, z, w)

in $\partial\mathbf{H}^2$. This is a complete invariant of the 4-tuple $(\phi(x), \phi(y), \phi(z), \phi(w))$ up to the symplectic imagegroup action, and allows to read informations on the translation lengths of the images $\rho(\gamma)$ of elements in the fundamental group from the values of the boundary map.

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