Kleinian groups Background

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Abstract We introduce basic notions about convex-cocompact actions on real rank one symmetric spaces. We focus mainly on the geometric interpretation as isometries of hyperbolic spaces, and we give the main characterizations of convex-cocompactness in this setting.

1 Introduction

Kleinian groups were introduced by Poincaré in the 1880's as groups of Möbius transformations of the complex plane acting discontinuously on some open domain of $\mathbb{C}$. He showed that any such transformation can be extended as an isometry of the hyperbolic space, establishing a link between conformal geometry and hyperbolic geometry. He understood that Kleinian groups were a good place to start investigating 3-dimensional hyperbolic geometry, but the necessary tools were not available at this time.

The study of Kleinian groups continued on the conformal side through the work of Ahlfors and Bers in the 1960's. One of the main important result obtained in this era is the Ahlfors finiteness theorem ([2]):

Theorem 1. *Let $\Gamma < \mathrm{PSL}(2,\mathbb{C})$ a finitely generated discrete group, with a non-empty domain of discontinuity Ω. Then Ω/Γ is a finite union of Riemann surfaces of finite type*

While the work of Ahlfors and Bers were very fruitful, not much was done on the hyperbolic side of the story. It's only with the so-called "Thurston's revolution" that the connection between Kleinian groups and hyperbolic 3-manifolds has been

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really exploited. The famous Hyperbolization of Haken manifold by Thurston uses extensively the theory of Kleinian groups to understand hyperbolic 3-manifolds.

In the last two decades, many progress were made in our understanding of Kleinian groups (as subgroups of $\mathrm{Isom}(\mathbb{H}^3)$), with the proofs of three major conjectures : The ending Lamination conjecture ([9, 5]), the tameness conjecture ([1, 6]) and the density conjecture ([4, 10, 11]).

The study of the generalizations of these deep results to higher dimensional contexts is relatively new. Given a Lie group G, one might ask what can be said about discrete subgroups of G. Such a study appears extremely rich and complicated. Even if one restricts to groups of isometries of hyperbolic n-space with $n > 3$, a picture as nice as in the $n = 3$ case does not occur (see Kapovich's survey [7]). In this context, it seems reasonable to restrict our study to discrete groups satisfying some additional hypothesis.

In this note, we mainly focus on convex-cocompact subgroups of Lie groups of real rank one. We will try to give many examples of such groups and try to explain the main properties of these groups. And we will see how one can generalize this notion to higher rank Lie groups.

2 Hyperbolic spaces

A (Riemannian) symmetric space is a Riemannian manifold X with the property that the geodesic reflection at any point is an isometry of X. In other words, for any $x \in X$ there is some isometry $s_x \in \mathrm{Isom(X)} = \mathrm{G}$ with the property $s_x(x) = x$ and $ds_x = -\mathrm{Id}$.

Such a symmetric space is geodesically complete and homogeneous. The stabilizer of a point p is can be denoted by K. One can identify X with the quotient G/K using the G-equivariant diffeomorphism $g \cdot K \mapsto g(p)$.

If $\gamma : \mathbb{R}^n \to X$ maps $\mathbb{R}^n$ locally isometrically into X, then it is called a flat subspace of X of dimension n. The real rank of X is the largest dimension of a flat subspace. This rank coincide with the real rank of the Lie group $G = \mathrm{Isom(X)}$.

A symmetric space is said to be of non-compact type if X has non-positive sectional curvature (but not identically 0).

Theorem 2. *Let X be a real rank 1 symmetric space, of non-compact type. Then X is one of the following : real hyperbolic space* $\mathbb{H}^n_{\mathbb{R}}$*, complex hyperbolic space* $\mathbb{H}^n_{\mathbb{C}}$*, quaternionic hyperbolic space* $\mathbb{H}^n_{\mathbb{Q}}$ *or the octonionic hyperbolic plane* $\mathbb{H}^2_{\mathbb{O}}$

We will now see a simple description of these hyperbolic spaces and their isometries.

2.1 Construction of hyperbolic spaces

Let $\mathbb{R}^{1,n}$ (resp. $\mathbb{C}^{1,n}$ and $\mathbb{Q}^{1,n}$) be the vector space over $\mathbb{R}$ (resp. $\mathbb{C}$ and $\mathbb{Q}$) with dimension $n+1$ equipped with a quadratic form (resp. Hermitian) of signature $(1,n)$, defined by:

$$\langle x,y\rangle = \overline{x_0}y_0 - \overline{x_1}y_1 - \cdots - \overline{x_n}y_n.$$

(For quaternions if $x = a+b\mathbf{i}+c\mathbf{j}+d\mathbf{k}$ then $\overline{x} = a-b\mathbf{i}-c\mathbf{j}-d\mathbf{k}$).

We can define the subsets :

$$V_+ = \{x \in \mathbb{R}^{1,n} \ : \ \langle x,x\rangle > 0\}$$
$$V_0 = \{x \in \mathbb{R}^{1,n} \setminus \{0\} \ : \ \langle x,x\rangle = 0\}$$

We define a right projection map $\mathbb{P} : \mathbb{R}^{1,n} \setminus \{0\} \to \mathbb{R}P^n$ (resp. to $\mathbb{C}\mathbb{P}^n$ and $\mathbb{Q}\mathbb{P}^n$) in the usual way. Then the corresponding hyperbolic space is $\mathbb{H}^n = \mathbb{P}V_+$. The metric on $\mathbb{H}^n$ is given by the distance function :

$$d(x,y) = \text{argcosh}\sqrt{\frac{\langle \mathrm{x},\mathrm{y}\rangle\langle \mathrm{y},\mathrm{x}\rangle}{\langle \mathrm{x},\mathrm{x}\rangle\langle \mathrm{y},\mathrm{y}\rangle}}$$

The boundary at infinity is $\partial\mathbb{H}^n = \mathbb{P}V_0$. We can also choose an affine chart with $x_0 = 1$ so that $\mathbb{H}^n$ is identified with the unit ball in $\mathbb{R}^n$ and the boundary $\partial\mathbb{H}^n$ is identified with the unit sphere in $\mathbb{R}^n$.

2.2 Isometries

The projective group of orthogonal matrices preserving the quadratic form is $\mathrm{PO}(1,\mathrm{n})$ (resp. $\mathrm{PU}(1,\mathrm{n})$ and $\mathrm{PSp}(1,\mathrm{n})$). Since the group $\mathrm{PO}(1,\mathrm{n})$ preserves the form. It is the group of isometries of the real hyperbolic space. (resp. complex and quaternionic)

For an element of $\gamma \in \mathrm{Isom}(\mathbb{H}^\mathrm{n})$, we define the displacement function :

$$d_\gamma(x) = d(x,\gamma(x)) \ x \in \mathbb{H}^n$$

Definition 1. A non-identity element $\gamma \in \mathrm{Isom}(\mathbb{H}^\mathrm{n})$ is said to be :

- *Hyperbolic* if the function d_γ is bounded away from zero. Its minimum is attained on a geodesic $A_\gamma \subset \mathbb{H}^n$ which is invariant under γ. The ideal end-points of this geodesic are the fixed point of γ in $\partial\mathbb{H}^n$.
- *parabolic* if the function d_γ is positive but has zero infimum on $\mathbb{H}^n$. There is exactly one fixed point in $\partial\mathbb{H}^n$.
- *elliptic* if $d_\gamma(x) = 0$ for some $x \in \mathbb{H}^n$.

3 Kleinian groups

3.1 definitions

A Kleinian group Γ is a discrete subgroup of $\mathrm{Isom}(\mathbb{H}^n)$.

Proposition 1. *A group Γ is kleinian if and only if Γ acts properly discontinuously on $\mathbb{H}^n$. Moreover, if Γ is torsion-free, then $\mathbb{H}^n/\Gamma$ is an hyperbolic n-manifold.*

Note that $\mathbb{H}^n/\Gamma$ has no reason to be compact or of finite volume. In general, such a manifold can be very wild, and that is why we restrict ourselves to groups that acts cocompactly on certain spaces.

Definition 2. A *lattice* is a subgroup $\Gamma \subset \mathrm{Isom}(\mathbb{H}^n)$ such that the quotient $\mathbb{H}^n/\Gamma$ is of finite volume. The lattice is called *uniform* if the quotient is compact (also called a cocompact lattice).

- The holonomy representation of an hyperbolic structure on a closed hyperbolic n-manifold is a uniform lattice in $\mathrm{Isom}(\mathbb{H}^n)$.
- the group $\mathrm{PSL}(2,\mathbb{Z}[i])$ is a lattice in $\mathrm{Isom}(\mathbb{H}^3)$ but is not uniform.

Definition 3. A kleinian group Γ is called *convex-cocompact* if there exists a closed convex invariant subset $\mathscr{C} \subset \mathbb{H}^n$ such that Γ acts cocompactly on $\mathscr{C}$.

Uniform lattices are convex-cocompact, with $\mathscr{C} = \mathbb{H}^n$. However, lattices that are not uniform are not convex cocompact, for example $\mathrm{PSL}(2,\mathbb{Z}) \subset \mathrm{PSL}(2,\mathbb{R}) = \mathrm{Isom}(\mathbb{H}^2)$

3.2 Action on the boundary

To study Kleinian groups and convex-cocompact groups, the action on the boundary $\partial\mathbb{H}^n$ of hyperbolic space is very important. Note that the group $\mathrm{Isom}(\mathbb{H}^n)$ is isomorphic to the group of Möbius transformations of the boundary $\mathrm{Mob}(\mathbb{S}^{n-1})$.

The boundary of hyperbolic space can be identified with G/P with P a minimal parabolic subgroup of the group of isometries. For a rank-one Lie group G, there is only one class of parabolic subgroup, and hence there is only one notion of boundary. For higher rank symmetric spaces, there are several possible choices, and hence one can consider different natural boundaries of a symmetric space.

Definition 4. The discontinuity set Ω_Γ is the set of all points of discontinuity, i.e.

$$\Omega_\Gamma = \{x \in \partial\mathbb{H}^n \mid \exists U \ni x, \sharp\{g \in \Gamma, gU \cap U \neq \emptyset\} < \infty\}$$

It's the largest open subset on which Γ acts properly discontinuously.

Definition 5. The limit set Λ_Γ is the set of accumulation points of an orbit $\overline{\Gamma \cdot o} \cap \partial\mathbb{H}^n$ for $o \in \mathbb{H}^n$.

The limit set does not depend on the choice of $o \in \mathbb{H}^n$. Note that as Γ acts properly discontinuously on $\mathbb{H}^n$, the accumulation points are necessarily in $\partial\mathbb{H}^n$. We have that $\partial\mathbb{H}^n = \Omega_\Gamma \sqcup \Lambda_\Gamma$.

The fixed points of an hyperbolic or parabolic element $\gamma \in \Gamma$ are in Λ_Γ.

Definition 6. A discrete group is *elementary* if $|\Lambda_\Gamma| < \infty$. In that case, $|\Lambda_\Gamma| \leq 2$.

If Γ is finite, then it is necessarily elementary as $\Lambda_\Gamma = \emptyset$. If $\Gamma = \langle\gamma\rangle$ then Γ is elementary.

Given a non-elementary group Γ, we can construct the convex hull $N(\Lambda_\Gamma)$ of Λ_Γ in $\mathbb{H}^n$. It is the smallest closed convex subset N of $\mathbb{H}^n$ such that

$$\overline{N(\Lambda_\Gamma)} \cap \partial\mathbb{H}^n = \Lambda_\Gamma$$

The set $N(\Lambda_\Gamma)$, is Γ-invariant, and the action of Γ on $N(\Lambda_\Gamma)$ is properly discontinuous. The quotient $N(\Lambda_\Gamma)/\Gamma \subset \mathbb{H}^n/\Gamma$ is called the convex core of $\mathbb{H}^n/\Gamma$,

Lemma 1. *Let* $\Gamma < \mathrm{Isom}(\mathbb{H}^n)$. *The following are equivalent :*

1. Γ *is convex-cocompact*
2. $N(\Lambda_\Gamma)/\Gamma$ *is compact.*
3. $(\mathbb{H}^n \cup \Omega_\Gamma)/\Gamma$ *is compact.*

Note that in general, it is not sufficient to have Ω_Γ/Γ compact to ensure that Γ is convex-cocompact.

4 Examples of convex-cocompact groups

4.1 Hyperbolic surfaces

If $\rho : \pi_1(S) \to \mathrm{PSL}(2,\mathbb{R})$ is the holonomy representation of an hyperbolic structure on a closed surface S, then $\Gamma = \rho(\pi_1(S))$ is a uniform lattice, and hence is convex-cocompact. The limit set is the full boundary $\partial\mathbb{H}^2$.

If the surface has finite volume with cusps (meaning curves around punctures are sent to parabolic elements), then the image of the holonomy representation is a lattice, but it's not convex-cocompact. The lifts of the cusps are dense in the boundary $\partial\mathbb{H}^2$ and hence the limit set is $\partial\mathbb{H}^2$.

If S is a compact surface with totally geodesic boundaries, then $\rho(\pi_1(S))$ is a subgroup of a uniform lattice. The lifts of the boundary components enclose a convex subset of $\mathbb{H}^2$, and the limit set is a Cantor dust in $\partial\mathbb{H}^2$. This is a particular case of a Schottky group.

More generally, given disjoint topological balls $D_1^-, D_1^+, D_2^-, \ldots D_k^+$, and maps $\phi_i \in \mathrm{Isom}(\mathbb{H}^n)$ such that $\phi_i(\mathbb{H}^n \setminus D_i^-) \subset D_i^+$ and $\phi_i^{-1}(\mathbb{H}^n \setminus D_i^+) \subset D_i^-$, then the group generated by the family ϕ_i is free in k generators, and is a a convex-cocompact group.

4.2 Arithmetic and non-arithmetic lattices

There are many examples of lattices as arithmetic lattices in Lie groups. Basically, an arithmetic group is a group that is algebraically isomorphic to something commensurable with $\mathrm{SL}(\mathrm{m},\mathbb{Z})$ for some $\mathbb{Z}$. The typical example is the modular group $\mathrm{PSL}(2,\mathbb{Z})$ which is a lattice in $\mathrm{Isom}(\mathbb{H}^2)$ whose quotient is the modular orbifold.

It is known that for quaternionic and octonionic spaces, $\mathbb{H}^n_{\mathbb{Q}}$ and $\mathbb{H}^2_{\mathbb{O}}$, every uniform lattice is arithmetic. The existence of non-arithmetic lattices is a difficult problem. In the real case, Gromov-Piateski-Shapiro construct examples of non-arithmetic lattices for all n (the existence of such groups is obvious for $n=2$).

For complex hyperbolic spaces, this is one of the main question still open. The existence of non-arithmetic lattices is known for $n=2$ and $n=3$, but is largely open for $n \geq 4$.

Note that in higher rank Lie group, a consequence of Margulis superrigidity is that all lattices are arithmetic.

4.3 Embedding and Deformations

A crucial property of convex-cocompact representations is the structural stability :

Theorem 3. *Let* $\rho_0 : \Gamma \to \mathrm{Isom}(\mathbb{H}^{\mathrm{n}})$ *be the inclusion map of a convex-cocompact group* Γ*. Then there exists an open neighborhood* $\mathscr{U} \subset \mathrm{Hom}(\Gamma, \mathrm{Isom}(\mathbb{H}^{\mathrm{n}}))$ *of* ρ_0 *such that for all* $\rho \in \mathscr{U}$*, the representation* ρ *is injective and* $\rho(\Gamma)$ *is convex-cocompact.*

In other words, the set of convex-cocompact representations of a given group Γ is open in the representation variety.

We can embed $\mathbb{H}^n$ into $\mathbb{H}^{n+1}$ as a totally geodesic subspace and hence, we can also embed a convex-cocompact subgroup of $\mathrm{Isom}(\mathbb{H}^{\mathrm{n}})$ into an higher dimensional hyperbolic space.

$$\Gamma \hookrightarrow \mathrm{Isom}(\mathbb{H}^{\mathrm{n}}) \hookrightarrow \mathrm{Isom}(\mathbb{H}^{\mathrm{n+1}})$$

Combining this with previous theorem, we can obtain large families of deformations of convex-cocompact groups.

4.4 Quasi-Fuchsian groups

The main examples of this construction are quasi-fuchsian groups. Using the identification $\mathrm{Isom}^+(\mathbb{H}^2) = \mathrm{PSL}(2,\mathbb{R})$ and $\mathrm{Isom}^+(\mathbb{H}^3) = \mathrm{PSL}(2,\mathbb{C})$. We have natural embedding of closed surface groups into $\mathrm{PSL}(2,\mathbb{C})$ and we denote by Γ the image of $\pi_1(S)$ in $\mathrm{PSL}(2,\mathbb{C})$.

In this case, the manifold $\mathbb{H}^3/\Gamma$ is homeomorphic $S \times \mathbb{R}$. The limit set of Γ is a circle in $\mathbb{S}^2$, and the discontinuity set $\Omega_\Gamma = \Omega_0 \cup \Omega_1$. The manifold $(\mathbb{H}^3 \cup \Omega_\Gamma)/\Gamma$ is homeomorphic to $S \times [0,1]$, and we have a conformal structure on Ω_i/Γ which is the conformal structure given by the embedding $\Gamma < \mathrm{Isom}(\mathbb{H}^2)$.

Definition 7. A group Γ is quasi-fuchsian if Λ_Γ is a topological circle.

Proposition 2. *Let Γ be a quasi-fuchsian group.*

- *Γ is quasi-conformally conjugate to a fuchsian group.*
- *$(\mathbb{H}^3 \cup \Omega_\Gamma)/\Gamma$ is homeomorphic to $S \times [0,1]$.*
- *We get two conformal structures on the boundaries Ω_i/Γ.*

Bers double uniformization theorem states that if X and Y are two conformal structures on S, then there exists a quasi-fuchsian group Γ such that $\Omega_0/\Gamma = X$ and $\Omega_1/\Gamma = Y$.

4.5 Complex quasi-fuchsian groups

We can also embed convex-cocompact group of $\mathrm{Isom}(\mathbb{H}^n_\mathbb{R})$ into $\mathrm{Isom}(\mathbb{H}^n_\mathbb{C})$. Indeed, we have a natural embedding $\mathrm{SO}(1,n)$ into $\mathrm{SU}(1,n)$. In the case of surface groups, we have two ways of embedding a fuchsian group $\Gamma < \mathrm{Isom}(\mathbb{H}^2)$ into $\mathrm{PU}(1,2)$. We say that a representation is $\mathbb{R}$-fuchsian (resp. $\mathbb{C}$-fuchsian) if it is obtained by the embedding $\mathrm{SO}(1,2) \hookrightarrow \mathrm{PU}(1,2)$ (resp. by the embedding $\mathrm{PU}(1,1) \hookrightarrow \mathrm{PU}(1,2)$)

The $\mathbb{C}$-fuchsian representations are "rigid", in the sense that the only deformation of such representations are compositions of deformations in $\mathrm{PU}(1,1)$ with conjugations. The deformation of $\mathbb{R}$-fuchsian are much more rich.

5 Generalizations of convex-cocompactness

We start with the following result :

Theorem 4 (Kleiner-Leeb, Quint). *Let G be a semisimple group of arbitrary rank, and Γ a Zariski-dense convex-cocompact subgroup of G. Then Γ is the product of convex-cocompact subgroups of simple factors of real rank 1 of G, and a uniform lattice in a product of the simple factors of rank greater than 2.*

To get interesting generalizations of the notion of convex-cocompactness to higher rank, one has to use other characterizations of convex-cocompactness.

5.1 Quasi-Isometric embedding

Let Γ be a finitely generated kleinian group and S a finite set of generators such that $S^{-1} = S$. Let d_S be the distance on Γ induced by this word-length.

Let $o \in x$ be a base point. The group Γ is quasi-isometrically embedded in Isom(X) if there exists constants (K, C) such that for all $\gamma \in \Gamma$

$$\frac{1}{K} d_X(\gamma \cdot o, o) - C \leq d_S(\gamma, e) \leq K d_X(\gamma \cdot o, o) + C$$

Theorem 5. *The group Γ is convex-cocompact if and only if it is quasi-isometrically embedded.*

5.2 Boundary Map

If the natural inclusion of Γ in G is a quasi-isometric embedding, then so is any orbit map $\Gamma \to X$. In particular, Γ is hyperbolic and any orbit map induces a continuous, injective, Γ-equivariant map

$$\xi : \partial\Gamma \to \partial X$$

Theorem 6. *A group Γ is convex-cocompact is and only if Γ is hyperbolic and there exists a continuous, injective, Γ-equivariant map $\xi : \partial\Gamma \to \Lambda_\Gamma$.*

5.3 Anosov Property

Let $\widetilde{U_0\Gamma} = (\partial\Gamma)^{(2)} \times \mathbb{R}$ with a flow $\{\widetilde{\phi}_t\}$ where $\mathbb{R}$ acts by translation on the real factor, and a metric such that

1. Γ acts cocompactly by isometries on $\widetilde{U_0\Gamma}$ and sends flow lines $(z^+, z^-) \times \mathbb{R}$ on flow lines $(\gamma \cdot z^+, \gamma \cdot z^-) \times \mathbb{R}$.
2. The $\mathbb{R}$ action is bi-lipschitz and commutes with the Γ action.
3. The maps $t \mapsto \widetilde{\phi}_t(y)$ are isometries for all $y \in \widetilde{U_0\Gamma}$.

The flow descends to a flow $\{\phi_t\}$ on $U_0\Gamma = \widetilde{U_0\Gamma}/\Gamma$. When Γ is the fundamental group of a closed hyperbolic manifold, we can choose $U_0\Gamma$ as the unit tangent bundle with ϕ_t the geodesic flow.

Let $\rho : \Gamma \to G = \mathrm{Isom}(\mathbb{H}^n)$ a representation, and let

$$\mathscr{X}_\rho = \widetilde{U_0\Gamma} \times (\partial\mathbb{H})^{(2)}/\Gamma$$

This space comes with a pair of transverse distributions $E^\pm$ coming from the transverse distributions on $(\partial\mathbb{H})^{(2)}$.

Definition 8. The representation ρ is Anosov if there exists a continuous section $\sigma : U_0\Gamma \to \mathscr{X}_\rho$ such that σ commutes with the flow, and the lifted action of $\{\phi_t\}$ on the pullback σ^*E^+ (resp. σ^*E^-) is contracting (resp. dilating).

Theorem 7. *A group $\Gamma < \mathrm{Isom}(\mathbb{H}^n)$ is convex-cocompact if and only if Γ is hyperbolic and the inclusion $\rho : \Gamma \to G$ is Anosov.*

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