Semisimple Lie algebras and applications to Symmetric spaces

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Abstract This set of notes was written for an introductory talk that the author gave at the *Workshop on* Sp$(4,\mathbb{R})$*-Anosov representations*. The main goal is to provide the reader with the relevant definitions, results and examples. Most of the proofs can be found in the standard references [4] and [5]. The sets of notes [3] and [6] were extremely useful in the process of writing what follows. We will firstly present the theory of (semisimple) Lie algebras, with special focus on the root space and the Cartan decomposition. We will then discuss some aspects of the theory of symmetric spaces.

1 Introduction

In this section we will recall some basic definitions and examples. It will be an occasion to fix notations.

Definition 1. A *Lie group* G is a real analytic manifold with a group structure such that the application $m(g,h) = g^{-1}h$, for $g,h \in G$, is real analytic. A Lie group is *complex* if it is a complex manifold and m is holomorphic. The *adjoint representation* of a Lie group G is defined as $\mathrm{Ad}(h) = d(C_h)_e \in \mathrm{GL}(T_eG)$ where e is the identity in G and C_h is conjugation by $h \in G$.

Example 1. Let V be a real vector space equipped with a symplectic form $\langle \cdot,\cdot \rangle$, i.e. a nondegenerate antisymmetric bilinear form. The *symplectic group* associated to this data is

$$\mathrm{Sp}(V) = \{g \in \mathrm{GL}(V) \colon \langle gv, gw \rangle = \langle v,w \rangle \text{ for all } v,w \in V\}.$$

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In particular if we fix an isomorphism $V \cong \mathbb{R}^{2n}$ and choose $\langle \cdot,\cdot \rangle$ defined by the matrix $J_{2n} = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$ we have $\mathrm{Sp}(2n,\mathbb{R}) = \{g \in \mathrm{GL}(2n,\mathbb{R}) \colon g^t J_{2n} g = J_{2n}\}$. Likewise we can define $\mathrm{Sp}(2n,\mathbb{C}) = \{g \in \mathrm{GL}(2n,\mathbb{C}) \colon g^t J_{2n} g = J_{2n}\}$ and we will denote with $\mathrm{Sp}(2n)$ the group $\mathrm{Sp}(2n,\mathbb{C}) \cap \mathrm{U}(2n)$.

A *Lie group morphism* is an immersion and a group homomorphism between two Lie groups.

Definition 2. Let F be a field. A *Lie algebra* $\mathfrak{g}$ is a (finite dimensional) F-vector space together with an F-bilinear map $[\cdot,\cdot]$ called *(Lie) bracket* satisfying:

1. $[x,x] = 0$ for all $x \in \mathfrak{g}$.
2. *(Jacobi identity)* $[[x,y],z] + [[y,z],x] + [[z,x],y] = 0$ for all $x,y,z \in \mathfrak{g}$.

The application $\mathrm{ad}(x) = [x,\cdot]$ is the *adjoint representation of* $\mathfrak{g}$.

Given a Lie group G we can naturally endow T_eG with a Lie algebra structure such that $\mathrm{ad} = (d\mathrm{Ad})_e$. We will call $\mathfrak{g} = T_eG$ the *Lie algebra of G* and it can also be identified with the set of left invariant vector fields on G. If G is a linear group, then $[\cdot,\cdot]$ coincides with the commutator.

Example 2. By differentiating the defining equation we have that the Lie algebra of $\mathrm{Sp}(2n,F)$ is $\mathfrak{sp}(2n,F) = \{x \in \mathfrak{gl}(2n,F) \colon x^t J_{2n} + J_{2n} x = 0\}$.

Propositions 1 and 2 motivate our focus on the study of Lie algebras and they will be used to carry information from Lie algebras to Lie groups. For proofs, see [6]. A *one parameter subgroup* of a Lie group G is a Lie group morphism $\gamma \colon \mathbb{R} \to G$.

Proposition 1. *The set $\mathscr{S}$ of one parameter subgroups of a group G is in bijection with the Lie algebra $\mathfrak{g}$ of G.*

Proposition 1 allows us to define the *exponential mapping*, $\exp \colon \mathfrak{g} \to G$ such that $\exp(x) = \gamma_x(1)$ where γ_x is the one parameter subgroup associated to $x \in \mathfrak{g}$.

Proposition 2. *Let G be a Lie group with Lie algebra $\mathfrak{g}$. There is a bijection between the set of Lie subalgebras of $\mathfrak{g}$ and the set of connected Lie subgroups of G.*

Proposition 2 allows us to define the *adjoint group* $\mathrm{Int}(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ as the subgroup of $\mathrm{GL}(\mathfrak{g})$ corresponding to $\mathrm{ad}(\mathfrak{g})$. If $\mathfrak{g}$ is the Lie algebra of a connected Lie group G, then $G/Z(G) \cong \mathrm{Ad}(G) \cong \mathrm{Int}(G)$ where $Z(G)$ is the center of G.

2 Semisimple Lie algebras

We will now start focussing on semisimple Lie algebras.

Definition 3. Let $\mathfrak{g}$ be a Lie algebra.

1. A vector subspace $\mathfrak{i} \subset \mathfrak{g}$ is an *ideal* if for all $x \in \mathfrak{i}$ and for all $y \in \mathfrak{g}$ we have $[x,y] \in \mathfrak{i}$.

2. $\mathfrak{g}$ is *abelian* if the Lie bracket on $\mathfrak{g}$ is trivial.
3. The Lie algebra $\mathfrak{g}$ is *simple* if it is not abelian and it has no nontrivial ideals.
4. A Lie algebra $\mathfrak{g}$ is *semisimple* if it can be written as a product of simple Lie algebras $\mathfrak{g} = \mathfrak{g}_1 \times \mathfrak{g}_2 \times \cdots \times \mathfrak{g}_k$.
5. A Lie algebra $\mathfrak{g}$ is *compact* if its adjoint group is compact.

Example 3. $\mathfrak{sl}(2,\mathbb{R}) \cong \mathfrak{sp}(2,\mathbb{R})$ is simple. $\mathfrak{u}(n)$ and $\mathfrak{sp}(n)$ are compact.

In general, it is not easy to detect compactness and semisimplicity.

Definition 4. For F a field, the *Killing form* on the F-Lie algebra $\mathfrak{g}$ is the F-bilinear form

$$\begin{aligned} B\colon \mathfrak{g}\times\mathfrak{g} &\to F \\ (x,y) &\mapsto \mathrm{tr}(\mathrm{ad}(x)\mathrm{ad}(y)) \end{aligned}$$

where tr is the trace.

Proposition 3. *Let $\mathfrak{g}$ be a Lie algebra and B its Killing form.*

1. *$\mathfrak{g}$ is semisimple if and only if B is nondegenerate.*
2. *If $\mathfrak{g}$ is semisimple, then $\mathfrak{g}$ is compact if and only if B is negative definite.*

Example 4. Choose the basis of $\mathfrak{g} = \mathfrak{sp}(4,\mathbb{C})$ (e_{ij} is the matrix with entry 1 in position i,j and 0 otherwise)

$$h_1 = e_{11} - e_{33} \quad h_2 = e_{22} - e_{44} \quad x_{12} = e_{12} - e_{43} \quad x_{21} = e_{21} - e_{34}$$
$$u_1 = e_{13} \quad u_2 = e_{e4} \quad z_{12} = e_{41} + e_{32} \quad v_1 = e_{31} \quad v_2 = e_{42} \quad y_{12} = e_{14} + e_{23}.$$

In this basis, the Killing form on $\mathfrak{g}$ is given by the matrix $B = 12(e_{11} + e_{22} + e_{34} + e_{43} + e_{69} + e_{96}) + 6(e_{58} + e_{7,10} + e_{85} + e_{10,7})$ whose eigenvalues are $-12, -6, 6, 12$. Thus, $\mathfrak{sp}(4,\mathbb{C})$ is semisimple.

3 Cartan subalgebras and Root space decomposition

We now introduce a decomposition of a complex semisimple Lie algebra that will be adapted to the real case in Section 5.

Definition 5. A *Cartan subalgebra* $\mathfrak{a}$ of a complex semisimple Lie algebra $\mathfrak{g}$ is a maximal abelian subalgebra such that all the elements in $\mathrm{ad}(\mathfrak{a})$ are diagonalizable.

Example 5. Diagonal matrices form a Cartan subalgebras in $\mathfrak{sl}(n,\mathbb{C})$ and $\mathfrak{sp}(2n,\mathbb{C})$.

In general, there is no reason for a Cartan subalgebra to exists. For instance, consider the algebra of strictly upper triangular matrices. However, if $\mathfrak{g}$ is a complex semisimple Lie algebra a Cartan subalgebra exists. *For the rest of this section $\mathfrak{g}$ will be a complex semisimple Lie algebra*.

Definition 6. Let $\mathfrak{a} \subset \mathfrak{g}$ be a Cartan subalgebra. For any $\alpha \in \mathfrak{a}^*$, let $\mathfrak{g}_\alpha$ be the vector space $\{x \in \mathfrak{g} \colon \mathrm{ad}(h)x = \alpha(h)x \text{ for all } h \in \mathfrak{a}\}$. An element α of the set

$$\Lambda = \{\beta \in \mathfrak{a}^* \colon \mathfrak{g}_\beta \neq 0\} \setminus \{0\}$$

is a *root* and $\mathfrak{g}_\alpha$ is its *root space*.

Example 6. Consider $\mathfrak{sp}(4,\mathbb{C})$ and fix $\mathfrak{a} = \mathrm{Span}(h_1,h_2)$. Let $\alpha_1, \alpha_2 \in \mathfrak{a}^*$ be defined by $\alpha_i(h_j) = \delta_{ij}$. The set of roots is $\Lambda = \{\pm(\alpha_1 - \alpha_2), \pm(\alpha_1 + \alpha_2), \pm 2\alpha_1, \pm 2\alpha_2\}$ with

$$\begin{gathered}
\mathfrak{g}_{\alpha_1-\alpha_2} = \mathrm{Span}(x_{12}),\ \mathfrak{g}_{-\alpha_1+\alpha_2} = \mathrm{Span}(x_{21}) \\
\mathfrak{g}_{\alpha_1+\alpha_2} = \mathrm{Span}(y_{12}),\ \mathfrak{g}_{-\alpha_1-\alpha_2} = \mathrm{Span}(z_{12}) \\
\mathfrak{g}_{2\alpha_i} = \mathrm{Span}(u_i),\ \mathfrak{g}_{-2\alpha_i} = \mathrm{Span}(v_i),\ i = 1,2.
\end{gathered}$$

and $\mathfrak{g}_0 = \alpha$. In particular $\mathfrak{sp}(4,\mathbb{C}) = \mathfrak{a} \oplus \bigoplus_{\alpha\in\Lambda} \mathfrak{g}_\alpha$.

For $\mathfrak{a}$ a Cartan subalgebra of a complex semisimple Lie algebra $\mathfrak{g}$ it is true in general that $\mathfrak{a} = \mathfrak{g}_0$.

Definition 7. The *root space decomposition* of a complex semisimple Lie algebra $\mathfrak{g}$ is the vector space decomposition

$$\mathfrak{g} = \mathfrak{a} \oplus \bigoplus_{\alpha\in\Lambda} \mathfrak{g}_\alpha.$$

We will now study some properties of this decomposition. Proofs can be found in [5] Section 8.3 and following.

Proposition 4. *Let $\mathfrak{g} = \mathfrak{a} \oplus \bigoplus_{\alpha\in\Lambda} \mathfrak{g}_\alpha$ be a root space decomposition.*

1. *For all $\alpha, \beta \in \mathfrak{a}^*$, $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$.*
2. *Λ spans $\mathfrak{a}^*$.*
3. *If $\alpha \in \Lambda$ then $-\alpha \in \Lambda$.*
4. *If x_α is a nonzero element in g_α there exists $y_\alpha \in \mathfrak{g}_{-\alpha}$ such that x_α, y_α and $h_\alpha = [x_\alpha, y_\alpha]$ span a Lie algebra isomorphic to $\mathfrak{sl}(2,\mathbb{C})$.*

Semisimplicity of $\mathfrak{g}$ and Proposition 4 imply that the restriction of the Killing form to $\mathfrak{a}$ is nondegenerate. Thus, $\mathfrak{a}^*$ can be identified to $\mathfrak{a}$ by associating $\alpha \in \mathfrak{a}^*$ to the element $t_\alpha \in \mathfrak{a}$ such that $B(t_\alpha, h) = \alpha(h)$. For α, β roots, set $(\alpha,\beta) = B(t_\alpha, t_\beta)$. The study of the representation theory of $\mathfrak{sl}(2,\mathbb{C})$ together with the last statement of Proposition 4 gives us the following additional properties of the root space decomposition.

Proposition 5. *Let α and β be roots.*

1. *The $\mathfrak{g}_\alpha$'s are 1-dimensional.*
2. *The only other scalar multiple of α in Λ is $-\alpha$.*
3. *$(\alpha,\beta) \in \mathbb{Q}$ and $\langle \beta, \alpha \rangle = \frac{2(\beta,\alpha)}{(\alpha,\alpha)} \in \mathbb{Z}$.*

4. *$\beta - \langle\beta,\alpha\rangle\alpha$ is a root.*

The previous propositions lead to the definition of *root system*. Let $(E,(\cdot,\cdot))$ be an Euclidean space and let $\sigma_\alpha(\beta) = \beta - \langle\beta,\alpha\rangle\alpha$ be the reflection associated to $\alpha \in E$.

Definition 8. $\Lambda \subset E \setminus \{0\}$ is a *root system* in E if it is finite, spans E and

1. If $\alpha \in \Lambda$, the only multiples of α in Λ are $\pm\alpha$.
2. If $\alpha \in \Lambda$, the reflection σ_α leaves Λ invariant.
3. If $\alpha, \beta \in \Lambda$, then $\langle\beta,\alpha\rangle \in \mathbb{Z}$.

To a root system Λ we can associate its *Weyl group* $W \subset \mathrm{GL}(E)$, the finite group generated by the σ_α's. A *Weyl chamber* is a connected component of the set $E \setminus (\cup_{\alpha\in\Lambda} w_\alpha)$, where w_α is the hyperplane of E associated to the reflection σ_α. The closure of a Weyl chamber is a fundamental domain for the action of the Weyl group.

Example 7. Let E be $\mathbb{R}^2$ with the standard inner product. If (e_1,e_2) is the standard basis of $\mathbb{R}^2$

$$\Lambda = \{\pm 2e_i, \pm(e_1+e_2), \pm(e_1-e_2)\}$$

can be seen as the root system corresponding to $\mathfrak{sp}(4,\mathbb{C})$. The Weyl group is isomporphic to $\mathbb{Z}/2\mathbb{Z} \rtimes (\mathbb{Z}/2\mathbb{Z})^2$.

The root space decomposition allows us to define some interesting subalgebras of $\mathfrak{g}$. First observe that, since in a root system the only scalar multiple of a root α is $-\alpha$, choosing a nontrivial linear functional $l\colon \mathfrak{a}^* \to \mathbb{R}$ gives a notion of positivity. The set of *positive roots* is $\Lambda^+ = \{\alpha \in \Lambda : l(\alpha) > 0\}$.

Definition 9. The *standard Borel subalgebra* (or *minimal parabolic*) associated to a choice of a Cartan subalgebra and a positivity notion is $\mathfrak{b} = \mathfrak{a} \oplus \bigoplus_{\alpha\in\Lambda^+} \mathfrak{g}_\alpha$. A *standard parabolic subalgebra* is any subalgebra containing $\mathfrak{b}$ and a *(generic) parabolic subalgebra* is a subalgebra conjugated to a standard parabolic subalgebra.

Example 8. In $\mathfrak{sp}(4,\mathbb{C})$ we have a natural positivity notion for which the standard Borel subalgebra is

$$\mathfrak{b} = \mathfrak{a} \oplus \mathfrak{g}_{\alpha_1-\alpha_2} \oplus \mathfrak{g}_{\alpha_1+\alpha_2} \oplus \mathfrak{g}_{2\alpha_1} \oplus \mathfrak{g}_{2\alpha_2}.$$

4 Cartan decomposition

We give a different natural vector space decomposition of a Lie algebra $\mathfrak{g}$ which will be very useful for the study Riemannian symmetric spaces of non-compact type. Throughout this section, given η an involution of $\mathfrak{g}$, we will denote B_η the bilinear map $-B(\cdot,\eta\cdot)$.

Definition 10. 1. Let $\mathfrak{h}$ be a real Lie algebra. Its *complexification* $\mathfrak{h}^{\mathbb{C}}$ is the complexification of $\mathfrak{h}$ as a vector space equipped with the Lie bracket

$$[x+iy, z+iw] = [x,z] - [y,w] + i([y,z]+[x,w]) \text{ for } x,y,z,w \in \mathfrak{h}.$$

2. Let $\mathfrak{g}$ be a complex Lie algebra and $\mathfrak{h}$ a real Lie subalgebra. $\mathfrak{h}$ is a *real form of* $\mathfrak{g}$ if there exists a $\mathbb{C}$-linear isomorphism $\phi\colon \mathfrak{h}^{\mathbb{C}} \to \mathfrak{g}$ such that $\phi_{|_{\mathfrak{h}}} = \mathrm{id}$.

Every complex semisimple Lie algebra has a compact real form. Moreover, we can choose one with special properties as described in the following theorem.

Theorem 1. *Let $\mathfrak{g}$ be a real semisimple Lie algebra, $\mathfrak{u}$ be a compact real form of $\mathfrak{g}^{\mathbb{C}}$. Let σ and τ be complex conjugation with respect to $\mathfrak{g}$ and $\mathfrak{u}$, respectively. There exists an automorphism $\psi\colon \mathfrak{g}^{\mathbb{C}} \to \mathfrak{g}^{\mathbb{C}}$ such that $\psi(\mathfrak{u})$ is invariant under σ.*

Definition 11. Let $\mathfrak{g}$ be a real Lie algebra and σ be complex conjugation of $\mathfrak{g}^{\mathbb{C}}$ with respect to $\mathfrak{g}$. A vector space decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ is a *Cartan decomposition* if there exists a compact real form $\mathfrak{u} \subset \mathfrak{g}^{\mathbb{C}}$ such that $\sigma(\mathfrak{u}) \subset \mathfrak{u}$, $\mathfrak{k} = \mathfrak{g} \cap \mathfrak{u}$ and $\mathfrak{p} = \mathfrak{g} \cap (i\mathfrak{u})$.

Proposition 6. *Let $\mathfrak{g}$ be a semisimple Lie algebra and let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be a Cartan decomposition.*

1. $[\mathfrak{k},\mathfrak{k}] \subset \mathfrak{k}$, $[\mathfrak{k},\mathfrak{p}] \subset \mathfrak{p}$ *and* $[\mathfrak{p},\mathfrak{p}] \subset \mathfrak{k}$.
2. $B_{|_{\mathfrak{k}\times\mathfrak{k}}}$ *is negative definite,* $B_{|_{\mathfrak{p}\times\mathfrak{p}}}$ *is positive definite and* $B(\mathfrak{k},\mathfrak{p}) = 0$.
3. *If* $\mathfrak{g} = \mathfrak{k}' + \mathfrak{p}'$ *is another Cartan decomposition, then there exists an automorphism* ϕ *such that* $\phi(\mathfrak{k}) = \mathfrak{k}'$ *and* $\phi(\mathfrak{p}) = \mathfrak{p}'$.
4. $\mathfrak{k}$ *is a maximal compact subalgebra of* $\mathfrak{g}$.

The following theorem, which characterizes Cartan decompositions, helps us get a better understanding of them.

Theorem 2. *Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be a vector space decomposition of a semisimple Lie algebra such that $\mathfrak{k} \subset \mathfrak{g}$ is a subalgebra. This is a Cartan decomposition if and only if the map $s\colon \mathfrak{k}+\mathfrak{p} \to \mathfrak{k}+\mathfrak{p}$, defined by $s(x,y) = (x,-y)$, is a Lie algebra automorphism and B_s is symmetric and positive definite.*

Definition 12. An involutive Lie algebra automorphism $\theta\colon \mathfrak{g} \to \mathfrak{g}$ is a *Cartan involution* if B_θ is symmetric and positive definite.

Example 9. The following example is from [1] Section 3. Consider the Lie algebras:

$$\mathfrak{c}_1(F) = \left\{ \begin{pmatrix} A & B \\ -B & A \end{pmatrix} : A,B \in \mathrm{Mat}_2(F) \text{ such that } A^t = -A,\ B^t = B \right\},$$
$$\mathfrak{c}_2(F) = \left\{ \begin{pmatrix} A & B \\ B & -A \end{pmatrix} : A,B \in \mathrm{Mat}_2(F) \text{ such that } A^t = A,\ B^t = B \right\}.$$

Notice $\mathfrak{u}(2) \cong \mathfrak{c}_1(\mathbb{R})$ via the map $A + iB \mapsto \begin{pmatrix} A & B \\ -B & A \end{pmatrix}$. We have that:

$$\mathfrak{sp}(4,\mathbb{C}) = \mathfrak{c}_1(\mathbb{C}) \oplus \mathfrak{c}_2(\mathbb{C}); \quad \mathfrak{sp}(4,\mathbb{R}) = \mathfrak{c}_1(\mathbb{R}) \oplus \mathfrak{c}_2(\mathbb{R}); \quad \mathfrak{sp}(4) = \mathfrak{c}_1(\mathbb{R}) \oplus i\mathfrak{c}_2(\mathbb{R}).$$

The above decomposition of $\mathfrak{sp}(4,\mathbb{R})$ is a Cartan decomposition with Cartan involution $\theta(x) = -x^t$.

5 Riemannian symmetric spaces

In this section we will start focussing on symmetric spaces following the exposition of [6]. The study of semisimple Lie algebras developped in the preceeding sections will be of great help.

Definition 13. A connected, (simply connected), real analytic Riemannian manifold M is a *Riemannian symmetric space* if each $p \in M$ is an isolated fixed point of an involutive isometry s_p of M.

A Riemannian symmetric space M is complete and the group $I(M)$ of isometries of M acts transitively on it. Equip $I(M)$ with the compact open topology.

Theorem 3. *$G = I_0(M)$ can be equipped with the structure of a Lie group and for all p, $K = \mathrm{Stab}_G(p)$ is a compact subgroup.*

Example 10. Let V be a real vector space with a symplectic form $\langle \cdot, \cdot \rangle$ and $\mathrm{Sp}(V)$ the associated symplectic group as in Example 1. The vector space V has even dimension since it admits a symplectic form so we can consider *complex structures* on it, i.e. $J \in \mathrm{GL}(V)$ such that $J^2 = -\mathrm{Id}_V$. Let $h_J(x,y) = \langle x, Jy \rangle + i\langle x, y \rangle$. Define

$$\chi = \{J \in \mathrm{GL}(V) \colon J \text{ is a complex structure on } V \text{ with } h_J \text{ Hermitian}\}.$$

It can be shown (see [2]) that χ is a Riemannian symmetric space with $G \cong \mathrm{Sp}(V)$ and $K \cong \mathrm{U}(n)$.

For every point p in M we have an automorphism of G given by $\sigma_p(g) = s_p g s_p$ and this defines an involution of the Lie algebra $\mathfrak{g}$ of G by setting $\theta_p = d(\sigma_p)_e$. Since θ_p is an involution, it is diagonalizable with eigenvalues ± 1.

Proposition 7. *1. $\mathfrak{k} = \{x \in \mathfrak{g} \colon \theta_p(x) = x\}$ is the Lie algebra of K.*
2. $\mathfrak{p} = \{x \in \mathfrak{g} \colon \theta_p(x) = -x\}$ is $\mathrm{Ad}(K)$ invariant.
3. The differential of the map $\pi_p \colon G \to M$ given by $\pi_p(g) = g \cdot p$ is surjective and has kernel $\mathfrak{k}$.
4. The geodesic rays in M starting at p are of the form $\gamma(t) = \gamma_x(t) \cdot p$ for some $x \in \mathfrak{p}$.

The goal for the rest of this section is to introduce the *restricted root space decomposition*. First, we need to recall a deep classification theorem for Riemannian symmetric spaces.

Definition 14. Let M be a Riemannian symmetric space

1. M is *irreducible* if it cannot be written as a nontrivial product of symmetric spaces.
2. M is of *non-compact type* if M has no Euclidean factors and non-positive sectional curvature.
3. M is of *compact type* if M has no Euclidean factors and non-negative sectional curvature.

Theorem 4. *Let M be a Riemannian symmetric space and $G = I_0(M)$.*

1. *M is of non-compact type if and only if G is semisimple and θ_p is a Cartan involution for all $p \in M$.*
2. *M is of compact type if and only if G is semisimple and compact.*
3. *$M = M_0 \times \cdots \times M_k$ with M_0 trivial or Euclidean and M_i is trivial or irreducible of compact or non-compact type for all $i = 1, \ldots, k$.*

It turns out that totally geodesic embedded subspaces, called *flats*, of Riemannian symmetric spaces detect many geometric properties of the space itself. In particular, the role of *maximal flats*, i.e. flats that are maximal with respect to inclusion, is key.

Proposition 8. *Let M be a symmetric space of non-compact type and let $G = I_0(M)$.*

1. *Let $p \in M$ and let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the vector space decomposition corresponding to p. The set of maximal abelian subalgebras $\mathfrak{a}$ in $\mathfrak{p}$ is in bijection with the set of maximal flats containing p by sending $\mathfrak{a}$ to $\exp \mathfrak{a} \cdot p$.*
2. *Every geodesic in M is contained in a maximal flat.*
3. *For any two maximal flats, there exists and isometry in G between them.*

Let M be a Riemannian symmetric space of non-compact type. Fix a Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ of the Lie algebra of $G = I_0(M)$ and let $\theta = \theta_p$ be the corresponding Cartan involution. A computation shows that $\mathrm{ad}(x)$ is self-adjoint with respect to B_θ for all $x \in \mathfrak{p}$. In particular, for every abelian subalgebra $\mathfrak{a} \subset \mathrm{p}$, we have that $\mathrm{ad}(\mathfrak{a})$ is simultaneously diagonalizable. Define for $\alpha \in \mathfrak{a}^*$

$$\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \colon \mathrm{ad}(h)x = \alpha(h)x \text{ for all } h \in \mathfrak{a}\}.$$

The set of *roots* Λ is the set of non-zero α for which $\mathfrak{g}_\alpha$ is non trivial. The *restricted root space decomposition* of $\mathfrak{g}$ is

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in \Lambda} \mathfrak{g}_\alpha$$

For the real case, some of the properties we discussed in 3 fail to hold. However, one can easily show that if $\alpha \in \Lambda$ then $-\alpha \in \Lambda$, $\theta \colon \mathfrak{g}_\alpha \to \mathfrak{g}_{-\alpha}$ is an isomorphism and that $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$. We can also give definitions of Borel and parabolic subalgebras in this context.

6 Hermitian Symmetric spaces

In this section we will follow [2] Section 2 to quickly discuss some extra structure for the symmetric space χ in Example 10.

Definition 15. An *almost complex structure* on a smooth manifold M of dimension n is a tensor such that for every $p \in M$ we have $J_p \colon T_pM \to T_pM$, $J_p^2 = -I_n$ and J_p varies smoothly with p.

Recall that a complex structure on a manifold induces an almost complex structure via multiplication by i.

Definition 16. Let M be a Riemannian symmetric space with Riemannian structure g. M is a *Hermitian symmetric space* if M admits a complex structure and $g(Jx,Jy) = g(x,y)$ for all $X,Y \in TM$.

For a Hermitian symmetric space we can define the two-form $\omega_M(x,y) = g(Jx,y)$ for $x,y \in TM$.

Lemma 1. *Let M be a Hermitian symmetric space with $G = I_0(M)$. Then, ω_M is G-invariant. Moreover, ω_M is closed.*

This lemma tells us that a Hermitian symmetric space is a Kähler manifold with Kähler form ω_M. This can be used, together with the fact that the center of $\mathrm{U}(n)$ has positive dimension, to show that χ is Hermitian. Lastly, we want to mention the following strong result about the topology of Hermitian symmetric spaces of non-compact type.

Theorem 5 (Harish-Chandra). *Any Hermitian symmetric space of non-compact type is biholomorphic to a bounded domain in a complex vector space.*

For an explicit realization of χ as a bounded domain in a complex vector space we refer to [2].

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