The connected components of the moduli space of maximal Sp $(4, \mathbb{R})$-representations

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Abstract This is a note from an expository talk given by the author at the "Workshop on Sp $(4, \mathbb{R})$-Anosov representations" which took place on January 10-18, 2016 in Granby, Colorado. We discuss the Higgs bundle techniques used in [2] in describing the connected components of the moduli space of maximal Sp $(4, \mathbb{R})$-representations and focus on a particular family of components with special significance, the ones which have recently come to be called the *Gothen components*.

1 Introduction

For a compact surface of genus $g \geq 2$, N. Hitchin in his seminal paper [7] constructed distinguished components of the moduli space of reductive surface group representations into the split real form G^r of any complex reductive Lie group. These components, known today as the *Hitchin components*, in the case when $G^r = \mathrm{Sp}(4, \mathbb{R})$ are all maximal and all contain representations $\rho : \pi_1(\Sigma) \to \mathrm{Sp}(4, \mathbb{R})$ which factor through the irreducible representation of $\mathrm{SL}(2, \mathbb{R})$ into $\mathrm{Sp}(4, \mathbb{R})$.

The component count for the character variety $\mathscr{R}^{\max}(\mathrm{Sp}(4, \mathbb{R}))$ and a careful description of the corresponding Higgs bundles in these components provide a better understanding of the representations which can not be deformed to representations which can factor through a proper reductive subgroup of $\mathrm{Sp}(4, \mathbb{R})$. The $2g-3$ many connected components of $\mathscr{R}^{\max}(\mathrm{Sp}(4, \mathbb{R}))$ which contain such representations are particularly interesting because:
i) Representations in these components all have Zariski dense image in $\mathrm{Sp}(4, \mathbb{R})$.
ii) The components are smooth, but unlike the Hitchin components, are topologically non-trivial.

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The present article is an exposition of the above results as these first appeared and discussed in detail by S. Bradlow, O. García-Prada and P. Gothen in [2] and the references therein.

2 Notation

Given a closed, connected, oriented surface Σ of genus $g \geq 2$, consider the *moduli space of reductive representations of* $\pi_1(\Sigma)$ *into* $\mathrm{Sp}(4,\mathbb{R})$ modulo conjugation, as the real-analytic variety

$$\mathscr{R}(\mathrm{Sp}(4,\mathbb{R})) = \mathrm{Hom}^+(\pi_1(\Sigma),\mathrm{Sp}(4,\mathbb{R}))/\mathrm{Sp}(4,\mathbb{R})$$

Fixing a complex structure, transforms our surface Σ into a Riemann surface X and introduces holomorphic techniques into the study of this moduli space. The non-abelian Hodge correspondence provides that a reductive representation $\rho : \pi_1(\Sigma) \to \mathrm{Sp}(4,\mathbb{R})$ corresponds to a polystable $\mathrm{Sp}(4,\mathbb{R})$-Higgs bundle over X defined by a triple (V,β,γ), where V is a rank 2 holomorphic vector bundle over X and $\beta : V^* \to V \otimes K,\ \gamma : V \to V^* \otimes K$ are symmetric homomorphisms, for a choice of a canonical line bundle K. The polystability condition for such triples was described by D. Alessandrini in this series of notes [1]. Letting $\mathscr{M}(\mathrm{Sp}(4,\mathbb{R}))$ denote the *moduli space of polystable* $\mathrm{Sp}(4,\mathbb{R})$-Higgs bundles modulo complex gauge transformations, the non-abelian Hodge correspondence provides an isomorphism of analytic varieties $\mathscr{M}(\mathrm{Sp}(4,\mathbb{R})) \simeq \mathscr{R}(\mathrm{Sp}(4,\mathbb{R}))$.

3 Toledo invariant

In this section we consider the basic topological invariant of an $\mathrm{Sp}(4,\mathbb{R})$-Higgs bundle and describe a sharp bound for it. The locally constant obstruction map

$$o_2 : \mathrm{Hom}(\pi_1(\Sigma),\mathrm{Sp}(4,\mathbb{R})) \to H^2(\Sigma,\pi_1(\mathrm{Sp}(4,\mathbb{R})))$$

is an integer valued function, since $H^2(\Sigma,\pi_1(\mathrm{Sp}(4,\mathbb{R}))) \simeq \pi_1(\mathrm{Sp}(4,\mathbb{R})) \simeq \mathbb{Z}$. Now, $o_2(\rho) = c_1(V)$, where V is the rank 2 vector bundle appearing in the Higgs bundle data (V,β,γ) corresponding to ρ. Thus, we have an integer valued function $d = \deg(V) = \langle c_1(V), [\Sigma] \rangle$, whose fibers are unions of connected components.

Definition 1. The *Toledo invariant* of the $\mathrm{Sp}(4,\mathbb{R})$-Higgs bundle (V,β,γ) is the integer

$$d = \deg(V)$$

We use the notation $\mathscr{M}_d = \mathscr{M}_d(\mathrm{Sp}(4,\mathbb{R}))$ to denote the moduli space parameterizing isomorphism classes of polystable $\mathrm{Sp}(4,\mathbb{R})$-Higgs bundles with $\deg(V) = d$.

Remark 1.

- For representations of $\pi_1(\Sigma)$ into $\mathrm{SL}(2,\mathbb{R}) \simeq \mathrm{Sp}(2,\mathbb{R})$ the Toledo invariant coincides with the Euler class of the corresponding flat $\mathrm{SL}(2,\mathbb{R})$-bundle. In this case the classical inequality of J. Milnor [8] provides an appropriate bound for this invariant:
$$|d| = |e(\rho)| \leq -\chi(\Sigma) = 2g-2$$
Later on, J. Wood [12] considered $\mathrm{SU}(1,1)$-bundles and so this is usually called the *Milnor-Wood inequality* in describing a sharp bound for the topological invariant, also for representations into more general Lie groups G.
- T. Hartnick and A. Ott describe in [6] how the generalized Milnor-Wood inequality of M. Burger and A. Iozzi translates under the non-abelian Hodge correspondence to an inequality for topological invariants of Higgs bundles.

The sharp bound below for the Toledo invariant for $G = \mathrm{Sp}(4,\mathbb{R})$ was first given by V. Turaev [11]. We will see however a proof by P. Gothen [5] in the Higgs bundle context, as this proof will be particularly motivating for what is going to follow.

Proposition 1. *(Milnor-Wood inequality) Let (V,β,γ) be a semistable $Sp(4,\mathbb{R})$-Higgs bundle. Then $|d| \leq 2g-2$.*

Remark 2.

- For this proof, we are considering the interpretation of the defining data for $\mathrm{Sp}(4,\mathbb{R})$-Higgs bundles as data for special $\mathrm{SL}(4,\mathbb{C})$-Higgs bundles, using the composite embedding $\mathrm{Sp}(4,\mathbb{R}) \hookrightarrow \mathrm{Sp}(4,\mathbb{C}) \hookrightarrow \mathrm{SL}(4,\mathbb{C})$. In this sense, an $\mathrm{Sp}(4,\mathbb{R})$-Higgs bundle (V,β,γ) is equivalent to the pair (E,Φ) where:
  1. $E = V \oplus V^*$ is a rank 4 holomorphic bundle over X.
  2. $\Phi : E \to E \otimes K$ with $\Phi = \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix}$.

  The Higgs bundle (E,Φ) is said to be *semistable* (resp. *stable*) if $\mu(F) \leq \mu(E)$ (resp. $\mu(F) < \mu(E)$), for every Φ-invariant proper subbundle $F \subset E$, i.e. such that $\Phi(F) \subset F \otimes K$. Moreover, it is said to be *polystable* if $E = \oplus E_i$, where each E_i is stable and of the same slope μ.
- The map $(V,\beta,\gamma) \mapsto (V^*,\gamma^t,\beta^t)$ provides an isomorphism $\mathscr{M}_d \simeq \mathscr{M}_{-d}$, thus we can restrict our attention to the case $d \geq 0$.

Proof. Let (E,Φ) with $E = V \oplus V^*$, $\Phi = \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix}$ be a semistable $\mathrm{Sp}(4,\mathbb{R})$-Higgs bundle and $d = \deg(V) \geq 0$. Then $\gamma \neq 0$, as otherwise V would be Φ-invariant and so would violate the stability condition, since

$$\mu(E) = \frac{\deg(E)}{rk(E)} = \frac{\deg(V \oplus V^*)}{rk(E)} = 0 \text{ and } \mu(V) = \frac{\deg(V)}{rk(V)} = \frac{d}{2} \geq 0$$

Consider the bundles $N = \ker(\gamma)$ and $I = \mathrm{Im}(\gamma) \otimes K^{-1} \leq V^*$.
We thus get an exact sequence of bundles

$$0 \to N \to V \to I \otimes K \to 0$$

and so

$$\begin{aligned} \deg(V) &= \deg(N) + \deg(I \otimes K) \\ &= \deg(N) + \deg(I) + rk(I)(2g-2) \end{aligned}$$

using that $\deg K = 2g - 2$.
Now, the bundles $N, V \oplus I \subset E$ are both Φ-invariant subbundles of E, thus from the semistability of (E, Φ) we get $\mu(N) \leq \mu(E)$ and $\mu(V \oplus I) \leq \mu(E)$. Therefore

$$\deg(N) \leq 0 \ \text{ and } d + \deg(I) \leq 0$$

We have also seen that

$$d = \deg(N) + \deg(I) + rk(I)(2g-2)$$

so from these relations we get

$$2d \leq rk(I)(2g-2)$$

and since $rk(I) = rk(\gamma) \leq 2$, we get the desired inequality.

Definition 2. We shall call $\mathrm{Sp}(4,\mathbb{R})$-Higgs bundles with Toledo invariant $d = 2g-2$ *maximal* and denote the components of $\mathscr{M}(\mathrm{Sp}(4,\mathbb{R}))$ with maximal positive Toledo invariant by $\mathscr{M}^{\max} \simeq \mathscr{M}_{2g-2}$.

4 Cayley partner

The Higgs bundle proof of proposition 1 opens the way to considering new topological invariants for our Higgs bundles in order to successfully count the components of $\mathscr{M}^{\max}$. We see from this proof that for a maximal semistable $\mathrm{Sp}(4,\mathbb{R})$-Higgs bundle (V, β, γ), the map $\gamma : V \to V^* \otimes K$ is an isomorphism. Moreover, since γ is symmetric, it equips V with a K-valued non-degenerate quadratic form.

Remark 3. Having considered $-(2g-2) \leq d \leq 0$ in the proof of proposition 1, then $\beta : V^* \to V \otimes K$ would be an isomorphism.

Now, fix a square root of the canonical bundle K, i.e. pick a line bundle L_0 such that $L_0^2 = K$ and define

$$W := V^* \otimes L_0$$

Then the map

$$q_W := \gamma \otimes I_{L_0^{-1}} : W^* \to W$$

defines a symmetric, non-degenerate form on W; in other words (W, q_W) *defines an* $O(2,\mathbb{C})$*-holomorphic bundle*. Moreover, the map β in (V, β, γ) defines a K^2-twisted endomorphism

$$\theta := \left(\gamma\otimes I_{K\otimes L_0}\right)\circ\left(\beta\circ I_{L_0}\right) : W \to W\otimes K^2$$

which is q_W-symmetric, i.e takes values in the isotropy representation for $GL(2,\mathbb{R})$. We say that (W,θ) defines a K^2-*twisted Higgs pair* with structure group $GL(2,\mathbb{R})$, i.e. θ takes values in $E\left(\mathfrak{m}^{\mathbb{C}}\right)\otimes K^2$.

Definition 3. We call (W,q_W,θ) the *Cayley partner* of the $\mathrm{Sp}(4,\mathbb{R})$-Higgs bundle (V,β,γ).

The original $\mathrm{Sp}(4,\mathbb{R})$-Higgs bundle data can clearly be recovered from the defining data for its Cayley partner, so the previous construction describes a well-defined correspondence $(V,\beta,\gamma)\mapsto(W,q_W,\theta)$. A careful comparison of the semistability condition for our maximal $\mathrm{Sp}(4,\mathbb{R})$-Higgs bundles (V,β,γ) and the one for their Cayley partners provides the following:

Theorem 1. *Let $\mathscr{M}^{\max}$ be the moduli space of polystable $Sp(4,\mathbb{R})$-Higgs bundles with degree $d=2g-2$ and let $\mathscr{M}'$ be the moduli space of polystable K^2-twisted $GL(2,\mathbb{R})$-Higgs pairs. The map $(V,\beta,\gamma)\mapsto(W,q_W,\theta)$ defines an isomorphism of complex algebraic varieties*

$$\mathscr{M}^{\max}\simeq\mathscr{M}'$$

Proof. see [3], theorem 4.3.

Remark 4. The theorem holds for polystable $\mathrm{Sp}(2n,\mathbb{R})$-Higgs bundles with $n\geq 2$ in general and the correspondence described is defined as the *Cayley correspondence*.

The Cayley correspondence brings in new topological invariants for our triples (V,β,γ), namely the first and second Stiefel-Whitney classes for the orthogonal bundle (W,q_W) underlying the Cayley partner:

$$w_1(W,q_W)\in H^1(X,\mathbb{Z}/2)\simeq(\mathbb{Z}/2)^{2g}$$

$$w_2(W,q_W)\in H^2(X,\mathbb{Z}/2)\simeq\mathbb{Z}/2$$

Therefore, we may define

$$w_i(V,\beta,\gamma):=w_i(W,q_W),\ i{=}1{,}2$$

and these invariants are well defined, because the Stiefel-Whitney classes are independent of the choice of the square root $L_0=K^{1/2}$ used in the definition of (W,q_W).

5 Component count for $\mathscr{M}^{\max}(\mathbf{Sp}(4,\mathbb{R}))$

In the previous section we have seen how $\mathrm{Sp}(4,\mathbb{R})$-Higgs bundles can be related to rank 2 orthogonal bundles, and the later were classified by D. Mumford in [9]. For our purposes we will be needing the following result from there:

Proposition 2. *Let (W, q_W) be a rank 2 orthogonal bundle. If $w_1(W, q_W) = 0$, then $W = L \oplus L^{-1}$, where L is a line bundle over X, and $q_W = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.*

Having this result in hand, we now obtain a first important description of the maximal semistable $\mathrm{Sp}(4,\mathbb{R})$-Higgs bundle data:

Proposition 3. *Let (V, β, γ) be a maximal semistable $Sp(4,\mathbb{R})$-Higgs bundle with $w_1(V, \beta, \gamma) = 0$ and let (W, q_W) be its Cayley partner, so $W = L \oplus L^{-1}$ and $q_W = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Then there is a line bundle N such that*

1. *$V = N \oplus N^{-1}K$ and with respect to this decomposition, the Higgs fields are $\beta = \begin{pmatrix} \beta_1 & \beta_3 \\ \beta_3 & \beta_2 \end{pmatrix} \in H^0(S^2V \otimes K)$ and $\gamma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in H^0(S^2V^* \otimes K)$*
2. *The degree of N is given by $\deg(N) = \deg(L) + g - 1$*
3. *The degree of L satisfies $0 \leq \deg(L) \leq 2g-2$ and for $\deg(L) > 0$, it is $\beta_2 \neq 0$.*
4. *When $\deg(L) > 0$, N is unique.*
   When $\deg(L) = 0$, N is unique up to multiplication by a square root of the trivial bundle.
   When $\deg(L) = 2g-2$, N satisfies $N^2 = K^3$.

Proof. (1) Consider $N := L \otimes L_0$. Then $V = W \otimes L_0 = (L \oplus L^{-1}) \otimes L_0 = N \oplus N^{-1}K$. Moreover, $\gamma = q \otimes I_{L_0} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} : (V^* \otimes L_0) \otimes L_0 \to (L_0^* \otimes V) \otimes L_0$ and since $\theta = (\gamma \otimes I_{K \otimes L_0}) \circ \beta \otimes I_{L_0}$ is q_W-symmetric, it turns out that $\beta = \begin{pmatrix} \beta_1 & \beta_3 \\ \beta_3 & \beta_2 \end{pmatrix} : V^* \to V \otimes K$.
(2) Since $N = L \otimes L_0$, then $\deg(N) = \deg(L_0) + \deg(L) = \deg(L) + g - 1$.
(3) Interchanging L with its dual if necessary we may assume that $\deg(L) \geq 0$. Now, whenever $\deg(L) > 0$, the Higgs field Φ must induce a non-zero holomorphic map $L \to L^{-1}K^2$ otherwise $L \subset W$ would violate the stability condition, since $\Phi : L \oplus L^{-1} \to (L \oplus L^{-1}) \otimes K^2 = LK^2 \oplus L^{-1}K^2$ and Φ should not preserve L. Hence global sections exist for the line bundle $L^{-2}K^2$, therefore $\deg(L^{-2}K^2) \geq 0$, i.e. $\deg(L) \leq 2g-2$. The fact that for $\deg(L) > 0$, β_2 is non-zero, follows also from the semistability condition.
(4) When $\deg(L) = 2g-2$, the Higgs field Φ induces a non-zero section of the degree 0 line bundle $L^{-2}K^2$, thus $L^2 = K^2$ and so $N^2 = (LL_0)^2 = K^3$.

Provoked by this proposition, we distinguish the Higgs bundles in $\mathscr{M}^{\max}$ in the following subfamilies:
(i) (V, β, γ) for which $w_1 \neq 0$.
(ii) (V, β, γ) for which $w_1 = 0$, and therefore $V = N \oplus N^{-1}K$ with $N := L \otimes L_0$ for $L_0 = K^{1/2}$ and $0 \leq \deg(L) \leq 2g-2$.
(iii) As a special case of (ii), (V, β, γ) with $\deg(L) = 2g-2$ in which case $N^2 = K^3$, thus such Higgs bundles are parameterized by spin structures $L_0 = K^{1/2}$ on the surface Σ underlying the Riemann surface X.
This motivates considering the following subspaces of the moduli space $\mathscr{M}^{\max}$ and we shall see next that these are actually connected components in $\mathscr{M}^{\max}$.

Definition 4. Let (V,β,γ) be a maximal $\mathrm{Sp}(4,\mathbb{R})$-Higgs bundle with topological invariants $w_1(W,q_W)\in H^1(X,\mathbb{Z}/2)\simeq(\mathbb{Z}/2)^{2g}$ and $w_2(W,q_W)\in H^2(X,\mathbb{Z}/2)\simeq\mathbb{Z}/2$. Define the following subspaces of $\mathscr{M}^{\max}$:

1. $\mathscr{M}_{w_1,w_2}=\{(V,\beta,\gamma)\,|\,w_1=w_1(V,\beta,\gamma)\neq 0,\ w_2=w_2(V,\beta,\gamma)\}/\simeq$
2. $\mathscr{M}_c^0=\{(V,\beta,\gamma)\,|\,w_1(V,\beta,\gamma)=0,\ c{:=}\deg(L)\}/\simeq$
3. $\mathscr{M}_{K^{1/2}}^T=\left\{(V,\beta,\gamma)\,\middle|\,V=N\oplus N^{-1}K \text{ with } N=K^{3/2}\right\}/\simeq$

where $\simeq$ indicates isomorphism classes of $\mathrm{Sp}(4,\mathbb{R})$-Higgs bundles.

Theorem 2 (P. Gothen). *The subspaces $\mathscr{M}_{w_1,w_2}$, $\mathscr{M}_c^0$, $\mathscr{M}_{K^{1/2}}^T$ are connected. Hence, $\mathscr{M}^{\max}$ decomposes in its connected components as*

$$\mathscr{M}^{\max}=\left(\bigcup_{w_1,w_2}\mathscr{M}_{w_1,w_2}\right)\bigcup\left(\bigcup_{0\le c<2g-2}\mathscr{M}_c^0\right)\bigcup\left(\bigcup_{K^{1/2}}\mathscr{M}_{K^{1/2}}^T\right)$$

and so the total number of connected components is $2\cdot\left(2^{2g}-1\right)+2g-2+2^{2g}=3\cdot 2^{2g}+2g-4$.

Remark 5. From N. Hitchin's fundamental paper [7], we knew already that there exists a distinguished component of $\mathscr{M}(\mathrm{Sp}(4,\mathbb{R}))$, the *Hitchin component*, isomorphic to a vector space and containing naturally the Teichmüller space. This theorem shows that there are exactly 2^{2g} such components, which are precisely the components $\mathscr{M}_{K^{1/2}}^T$.

Proof. We treat each case separately:
(i) $\mathscr{M}_{K^{1/2}}^T$ is connected. The Cayley partner (W,q_W) of a Higgs bundle $(V,\beta,\gamma)\in\mathscr{M}_{K^{1/2}}^T$ is completely determined by the line bundle L in the decomposition $W=L\oplus L^{-1}$. But here $L=K^{1/2}$ and every (W,q_W) is stable. Hence,

$$\mathscr{M}_{K^{1/2}}^T\simeq H^0\left(\Sigma,\mathrm{End}(W)\otimes K^2\right)$$

and these Higgs field are q_W-symmetric, i.e. $\Phi=\begin{pmatrix}\Phi_{11}&\Phi_{12}\\ \Phi_{12}&\Phi_{22}\end{pmatrix}$. Therefore $\mathscr{M}_{K^{1/2}}^T$ is isomorphic to the vector space $H^0\left(\Sigma,K^2\right)\oplus H^0\left(\Sigma,K^2\right)\oplus H^0\left(\Sigma,K^4\right)$.
(ii) $\mathscr{M}_c^0$ is connected. The proof is based on the study of the local minima of the proper Hitchin map on $\mathscr{M}_c^0$.
For $c>0$, the Higgs field Φ must be non-zero, otherwise the subbundle $L\subset W$ for the Cayley partner (W,q_W) would violate the stability condition. Moreover, for the critical points in $\mathscr{M}_c^0$, $\Phi=\begin{pmatrix}0&0\\ \tilde{\Phi}&0\end{pmatrix}$ with $\tilde{\Phi}\in H^0\left(\Sigma,L^{-2}K^2\right)$. Now, the subspace of local minima $\mathscr{N}_c^0\subset\mathscr{M}_c^0$ fits into the pullback diagram

$$\begin{array}{ccc} \mathcal{N}_c^0 & \longrightarrow & \mathrm{Jac}^c(\Sigma) \\ \Big\downarrow \pi & & \Big\downarrow L \to L^{-2}K^2 \\ S^{4g-4-2c}\Sigma & \xrightarrow{D\to[D]} & \mathrm{Jac}^{4g-4-2c}(\Sigma) \end{array}$$

where $\pi(W, q_W, \Phi) = (\Phi)$.
Thus, $\mathcal{N}_c^0$ is connected, so from the properness of the Hitchin map $f : \mathcal{M}_c^0 \to \mathbb{R}$, it follows that $\mathcal{M}_c^0$ is connected, for $c > 0$.
For $c = 0$, every local minimum of f on $\mathcal{M}_c^0$ has $\Phi = 0$, so the subspace of local minima is isomorphic to the moduli space of polystable (W, q_W), where $W = L \oplus L^{-1}$ with $\deg(L) = 0$. It follows that there is a surjective continuous map $\mathrm{Jac}^0(\Sigma) \to \mathcal{N}_0^0$, with $L \mapsto (W, q_W)$, and so $\mathcal{N}_0^0$ is connected.
(iii) $\mathcal{M}_{w_1,w_2}$ is connected. This is the hardest part and we shall include here just a sketch; for the complete proof see theorem 5.8 in [5].
Similarly to the previous part, we are trying to show that the subspace of local minima of the Hitchin map $\mathcal{N}_{w_1,w_2} \subset \mathcal{M}_{w_1,w_2}$ is connected. These subspaces consist of critical points (V, β, γ) with $\beta = 0$ and $\gamma \neq 0$. There is a connected double cover $\tilde{\Sigma} \to \Sigma$ given by $w_1 \in H^1(\Sigma, \mathbb{Z}/2)$. Then it turns out that $\mathcal{N}_{w_1,0} \cup \mathcal{N}_{w_1,1} = \ker(1 + \tau^*)$, where $\tau : \tilde{\Sigma} \to \tilde{\Sigma}$ is the involution interchanging the sheets of the covering.
Now, $\ker(1 + \tau^*) = P^+ \cup P^-$ where the two components P^+ and P^- are the abelian varieties associated to the double cover of Σ given by w_1, each of them a translate of the Prym variety of the covering. Then $\mathcal{N}_{w_1,0} \cup \mathcal{N}_{w_1,1} = P^+ \cup P^-$, hence $\mathcal{N}_{w_1,w_2}$ is connected.

6 Higgs bundle parametrization of the components of $\mathcal{M}^{\max}$

The description of a maximal $\mathrm{Sp}(4,\mathbb{R})$-Higgs bundle from the data for its Cayley partner, as well as proposition 3 and theorem 2, provide a description of the $\mathrm{Sp}(4,\mathbb{R})$-Higgs bundle data in each connected component of $\mathcal{M}^{\max}$. This information is summarized in the following table:

Component	V	β	γ
$\mathcal{M}^T_{K^{1/2}}$	$K^{3/2} \oplus K^{-1/2}$	$\begin{pmatrix} \beta_1 & \beta_3 \\ \beta_3 & 1 \end{pmatrix}, \begin{cases} \beta_3 \in H^0(K^2) \\ \beta_1 = \text{const.}(\beta_3)^2 \end{cases}$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
$\mathcal{M}_c^0$	$V = N \oplus N^{-1}K$, with $g - 1 < \deg(N) < 3g - 3$	$\begin{pmatrix} \beta_1 & \beta_3 \\ \beta_3 & \beta_2 \end{pmatrix}$, with $\beta_2 \neq 0$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
$\mathcal{M}_0^0$	$V = N \oplus N^{-1}K$, with $\deg(N) = g - 1$	$\begin{pmatrix} \beta_1 & \beta_3 \\ \beta_3 & \beta_2 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
$\mathcal{M}_{w_1,w_2}$	$V = W \otimes L_0$, with $L_0^2 = K$	$\beta \in H^0(S^2V \otimes K)$	$\gamma = q_W \otimes I_{L_0}$

7 The corresponding maximal fundamental group representations into $\mathbf{Sp}(4,\mathbb{R})$

So far we have been interested in identifying particular polystable $\mathrm{Sp}(4,\mathbb{R})$-Higgs bundles in the connected components of $\mathscr{M}^{\max}$. The non-abelian Hodge theorem provides a homeomorphism

$$\mathscr{R}^{\max} \simeq \mathscr{M}^{\max}$$

and we analogously consider the following subspaces of $\mathscr{R}^{\max}$

$$\mathscr{R}_{w_1,w_2} \simeq \mathscr{M}_{w_1,w_2},\ \mathscr{R}_c^0 \simeq \mathscr{M}_c^0,\ \mathscr{R}_{K^{1/2}}^T \simeq \mathscr{M}_{K^{1/2}}^T$$

which are furthermore connected components in $\mathscr{R}^{\max}$.

Now, the possible subgroups of $\mathrm{Sp}(4,\mathbb{R})$ through which a maximal representation $\rho : \pi_1(\Sigma) \to \mathrm{Sp}(4,\mathbb{R})$ can factor, can be explicitly described:

Proposition 4. *Let $\rho : \pi_1(\Sigma) \to Sp(4,\mathbb{R})$ be maximal and assume that ρ factors through a proper reductive subgroup $\tilde{G} \subset Sp(4,\mathbb{R})$. Then, up to conjugation, the group $\tilde{G}$ is contained in one of the subgroups G_i, G_Δ and G_p, where*

1. *G_i, the normalizer of the irreducible four-dimensional representation of $SL(2,\mathbb{R})$ into $Sp(4,\mathbb{R})$.*
2. *G_p, the normalizer of the product representation $\rho_p : SL(2,\mathbb{R}) \times SL(2,\mathbb{R}) \to Sp(4,\mathbb{R})$*
3. *G_Δ, the normalizer of the composition of ρ_p with the diagonal embedding of $SL(2,\mathbb{R})$ into $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$.*

Proof. see B. Pozzetti [10] in this series of notes.

Defining the group $\mathrm{Sp}(4,\mathbb{R})$ with respect to the symplectic form $J_{12} = \begin{pmatrix} J & 0 \\ 0 & J \end{pmatrix}$, where $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, explicit calculations now show:

1. $G_i = \mathrm{SL}(2,\mathbb{R})$
2. $G_p = \left\{ \begin{pmatrix} X & Y \\ Z & T \end{pmatrix} \in \mathrm{Sp}(4,\mathbb{R}) \,|\, either\ Y = Z = 0\ or\ X = T = 0 \right\}$
3. $G_\Delta = \left\{ \begin{pmatrix} xA & yA \\ zA & tA \end{pmatrix} \middle| X = \begin{pmatrix} x & y \\ z & t \end{pmatrix} \in \mathrm{O}(2) \text{ and } A \in \mathrm{SL}(2,\mathbb{R}) \right\} = \mathrm{O}(2) \otimes \mathrm{SL}(2,\mathbb{R})$

The goal in this section is to identify in which connected components of $\mathscr{R}^{\max}$ we can find representations that can factor through one of the subgroups G_i, G_Δ or G_p described above. According to the non-abelian Hodge correspondence, a reductive representation $\rho : \pi_1(\Sigma) \to \mathrm{Sp}(4,\mathbb{R})$ that factors through a proper reductive subgroup $G_* \subset \mathrm{Sp}(4,\mathbb{R})$ corresponds to a polystable $\mathrm{Sp}(4,\mathbb{R})$-Higgs bundle (V,β,γ) for which the structure group reduces to G_* in the following sense:

Definition 5. For a real reductive subgroup $G_* \subset \mathrm{Sp}(4,\mathbb{R})$ and (E,Φ) an $\mathrm{Sp}(4,\mathbb{R})$-Higgs bundle, a reduction of (E,Φ) to a G_*-Higgs bundle (E^*,Φ^*) is given by the following data:

1. a holomorphic reduction of the structure group of E to a principal $H^{*\mathbb{C}}$-bundle $E^* \hookrightarrow E$; equivalently, this is given by a holomorphic section σ of $E \times_{H^{\mathbb{C}}} (H^{\mathbb{C}}/H^{*\mathbb{C}})$
2. a holomorphic section Φ^* of $E^*(\mathfrak{m}^{*\mathbb{C}}) \otimes K$, which maps to Φ under the embedding $E^*(\mathfrak{m}^{*\mathbb{C}}) \otimes K \to E(\mathfrak{m}^{\mathbb{C}}) \otimes K$.

Furthermore, it is important to note that if $(E_{H^{\mathbb{C}}},\Phi)$ is polystable as an $\mathrm{Sp}(4,\mathbb{R})$-Higgs bundle, then $\left(E_{H^{*\mathbb{C}}},\Phi^*\right)$ is polystable as a G_*-Higgs bundle (see the discussion in section 2.3 in [2]).

To get to the desired identification we mentioned earlier, for each of the possible reductive subgroups $G_* \subset \mathrm{Sp}(4,\mathbb{R})$, we first need to describe the defining data for the G_*-Higgs bundles, then describe the semistable $\mathrm{Sp}(4,\mathbb{R})$-Higgs bundles for which the structure group reduces to G_* and lastly, using the information from the table in section 6 we can see where these should lie.

Eventually, we get the following picture for the $3 \cdot 2^{2g} + 2g - 4$ many connected components of $\mathscr{M}^{\max}$, regarding particular fundamental group representations that should lie in these components (for details, see sections 6-8 in [2]):

- <u>2^{2g} *Hitchin components* $\mathscr{M}^{T}_{K^{1/2}}$</u>

$$\left\{\begin{array}{c}\mathrm{Sp}(4,\mathbb{R})-\text{Higgs bundles}\\ \text{str.gp. reduces to } \mathrm{SL}(2,\mathbb{R})\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\rho:\pi_1(\Sigma)\to\mathrm{Sp}(4,\mathbb{R})\\ \text{factors through } \mathrm{SL}(2,\mathbb{R})\end{array}\right\}$$

- <u>$2\cdot 2^{2g}-1$ *components* $\mathscr{M}_{w_1,w_2}, \mathscr{M}^0_0$</u>

$$\left\{\begin{array}{c}\mathrm{Sp}(4,\mathbb{R})-\text{Higgs bundles}\\ \text{str. gp. reduces to } G_p\\ \text{and}\\ \mathrm{Sp}(4,\mathbb{R})-\text{Higgs bundles}\\ \text{str. gp. reduces to } G_\Delta\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\rho:\pi_1(\Sigma)\to\mathrm{Sp}(4,\mathbb{R})\\ \text{factors through } G_p\\ \text{and}\\ \rho:\pi_1(\Sigma)\to\mathrm{Sp}(4,\mathbb{R})\\ \text{factors through } G_\Delta\end{array}\right\}$$

- <u>$2g-3$ *components* $\mathscr{M}^0_c$</u>

$$\left\{\begin{array}{c}\mathrm{Sp}(4,\mathbb{R})-\text{Higgs bundles}\\ \text{str.gp. does not reduce}\\ \text{to any } G_* \subset \mathrm{Sp}(4,\mathbb{R})\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\rho:\pi_1(\Sigma)\to\mathrm{Sp}(4,\mathbb{R})\\ \text{does not factor}\\ \text{through any } G_* \subset \mathrm{Sp}(4,\mathbb{R})\end{array}\right\}$$

8 The $2g-3$ exceptional components of $\mathscr{M}^{\max}$

From the investigation summarized in the previous section we deduce that:

Theorem 3. *Among the $3\cdot 2^{2g}+2g-4$ connected components of $\mathscr{M}^{\max}\simeq\mathscr{R}^{\max}$, there are $2g-3$ components where the corresponding Higgs bundles do not admit a reduction of structure group to any proper reductive subgroup of $Sp(4,\mathbb{R})$. Equivalently, the corresponding representations do not factor through any proper reductive subgroup of $Sp(4,\mathbb{R})$, thus they have Zariski-dense image in $Sp(4,\mathbb{R})$.*

Definition 6. We shall call the components described in the previous theorem the *Gothen components* of $\mathscr{M}^{\max}$.

Remark 6. Quite differently than the $\mathrm{Sp}(4,\mathbb{R})$-case, the moduli space of maximal polystable $\mathrm{Sp}(2n,\mathbb{R})$-Higgs bundles has $3\cdot 2^{2g}$ many connected components for every $n\geq 3$, and any $\mathrm{Sp}(2n,\mathbb{R})$-Higgs bundle in those can be deformed to a G_*-Higgs bundle for some proper reductive Zariski closed subgroup $G_*\subset\mathrm{Sp}(2n,\mathbb{R})$ (see section 9 in [2] and section 8 in [3] for details).

Let (V,β,γ) be a maximal semistable $\mathrm{Sp}(4,\mathbb{R})$-Higgs bundle in the Gothen components. Next we collect some results concerning such Higgs bundles, the first three of which we have already seen.

1. For the Cayley partner $\left(W=L\oplus L^{-1}, q_W=\begin{pmatrix}0&1\\1&0\end{pmatrix}\right)$ the first Stiefel-Whitney class vanishes: $w_1(V,\beta,\gamma)=w_1(W,q_W)=0$, where L is a line bundle on X.
2. The bundle V decomposes as $V=N\oplus N^{-1}K$, for a line bundle N with $\deg(N)=\deg(L)+g-1$ and $g-1<\deg(N)<3g-3$, in other words $0<\deg(L)<2g-2$.
3. The Higgs fields with respect to this decomposition for V are $\beta=\begin{pmatrix}\beta_1&\beta_3\\\beta_3&\beta_2\end{pmatrix}\in H^0\left(S^2V\otimes K\right)$, with $\beta_2\neq 0$ and $\gamma=\begin{pmatrix}0&1\\1&0\end{pmatrix}\in H^0\left(S^2V^*\otimes K\right)$.
4. Furthermore, since $0<\deg(L)<2g-2=\deg(V)$, all points in the Gothen components are represented by stable $\mathrm{Sp}(4,\mathbb{R})$-Higgs bundles. From this fact, it follows that these Higgs bundles are smooth points in the moduli space. This is proven using the standard slice method construction used to prove that the moduli space $\mathscr{M}_d(G)$ has the structure of a complex analytic variety (see [4], prop. 3.18 and the discussion preceding this). Hence, *the Gothen components are smooth.*

   Remark 7. Using these same arguments, one shows that all $\mathrm{Sp}(4,\mathbb{R})$-Higgs bundles in the 2^{2g}-many Hitchin components $\mathscr{M}^T_{K^{1/2}}$ are stable with $\beta_2\neq 0$, and smooth as well.

5. Isomorphism classes of $\mathrm{Sp}(4,\mathbb{R})$-Higgs bundles in the Gothen components can be also described. Considering a representative of such an isomorphism class to be determined by a triple $(N,\beta_1,\beta_2,\beta_3)$, the following holds (see [2], prop. 3.28):

Proposition 5. *Fix* $c = \deg(L)$ *with* $0 < c < 2g-2$. *Tuples* $(N, \beta_1, \beta_2, \beta_3)$ *and* $(N', \beta'_1, \beta'_2, \beta'_3)$ *define the same isomorphism class in* $\mathscr{M}_c^0$ *if and only if* $N = N'$ *and* $(\beta'_1, \beta'_2, \beta'_3) = \left(t^2\beta_1, t^{-2}\beta_2, \beta_3\right)$, *for some* $t \in \mathbb{C}^*$.

6. Lastly, there is a fibration of a certain subfamily of the Gothen components over the Jacobian Jac^d of degree d line bundles on X (see [2], prop. 3.30):

Proposition 6. *For* $0 < c < g-1$, *the space* $\mathscr{M}_c^0$ *fibers over* Jac^d *with* $d = c + g - 1$, *and the fibers are given by*

$$\mathscr{F}^d = \left[\left(\mathbb{C}^r \oplus (\mathbb{C}^*)^{s+1}\right)/\mathbb{C}^*\right] \times \mathbb{C}^{3g-3}$$

where $r = 2c + 3g - 3$, $s = 3g - 4 - 2c$ *and the* $\mathbb{C}^*$*-action is given by the relation* $t(\mathbf{z}, \mathbf{w}) = \left(t^2\mathbf{z}, t^{-2}\mathbf{w}\right)$.

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