Anosov representations via actions on boundaries and domains of discontinuity
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Abstract We summarize work of Guéritaud–Guichard–Kassel–Wienhard [8] characterizing Anosov representations of a hyperbolic group Γ into a semisimple Lie group G in terms of the Cartan or Lyapunov projections of G. We give applications to properly discontinuous actions on homogeneous spaces G/H.

1 Brief overview of Anosov representations

Let G be a semisimple Lie group and P, P^- two opposite parabolic subgroups of G.

Example 1.1. For $G = \mathrm{Sp}(4,\mathbf{R})$, stabilizer of the symplectic form $x_1 \wedge x_3 + x_2 \wedge x_4$ on $\mathbf{R}^4$, we may take P (resp. P^-) to be the stabilizer of the line $\mathbf{R}e_1$ (resp. $\mathbf{R}e_3$).

In 2006, Labourie [17] introduced the notion of a $(G, G/(P \cap P^-))$-Anosov structure on a manifold M equipped with an Anosov flow $(\varphi_t)_{t\in\mathbf{R}}$, as a dynamical analogue of a geometric structure on M. Such a structure comes with a holonomy representation $\rho : \Gamma \to G$ called a *P-Anosov representation*, where Γ is a word hyperbolic group. Labourie proved that Anosov representations are quasi-isometric embeddings (in particular, they have a finite kernel and a discrete image), and that they are structurally stable (which is nice, since the class of quasi-isometric embeddings $\Gamma \to G$ is not structurally stable when G has higher real rank — see Guichard's thesis or the appendix of [8]).

For $G = \mathrm{SL}_n(\mathbf{R})$ and Γ a closed surface group, Labourie showed that the *Hitchin component* of $\mathrm{Hom}(\Gamma, G)$ (i.e. the connected component containing the Teichmüller space of Γ via the irreducible representation $\mathrm{PSL}_2(\mathbf{R}) \hookrightarrow G$) consists entirely of Anosov representations. Together with fundamental work of Hitchin [12] and Fock–Goncharov [7], this showed that the Hitchin component shares some striking

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common features with Teichmüller space — a starting point to *Higher Teichmüller theory*.

Anosov representations were later studied by Guichard–Wienhard [10], who unified and developed further the theory. In addition to Hitchin components, important examples include:

- convex cocompact representations into G of real rank one;
- maximal representations into G of Hermitian type (see Beatrice's talk);
- representations into $G = \mathrm{PGL}_n(\mathbf{R})$ associated with certain convex divisible sets in real projective geometry, as introduced by Benoist [5];
- the natural inclusion of Schottky groups, playing ping pong on G/P, as introduced by Benoist [3];
- all deformations (e.g. by bending) into $G = \mathrm{SO}(n,2)$ of a uniform lattice Γ of $\mathrm{SO}(n,1)$ (see Barbot [1], Barbot–Mérigot [2]).

Recent work of Kapovich–Leeb–Porti [14, 15, 16, 13] shows that Anosov representations are good analogues of convex cocompact representations in higher real rank; this will be the subject of Jeff's talk. In the current talk we shall see some independent results of [8, 9] in this direction, including:

- characterizations of Anosov representations in terms of growth properties of the Cartan or Lyapunov projections (Theorems 5.1 and 5.2);
- construction of compactifications of Riemannian locally symmetric spaces arising from Anosov representations (Section 8).

We shall actually see results of [8, 9] applying to much more general discrete subgroups of G than images of Anosov representations, including:

- existence and point-by-point construction of a continuous equivariant boundary map (Theorem 4.1);
- construction of domains of discontinuity containing the Riemannian symmetric space G/K in some flag variety G'/P' for *any* discrete subgroup of G (Section 8).

One important motivation for [8] was the study of properly discontinuous actions on *non-Riemannian* homogeneous spaces: this will be the subject of Section 7. The general approach will consist in studying the dynamics of discrete groups acting on flag varieties, and a crucial role will be played throughout by the Cartan projection.

2 Notation

In the whole talk we fix a noncompact, linear, real, semisimple (or more generally reductive) Lie group G; for simplicity we assume G to be connected.

Recall that G admits the Cartan decomposition $G = K(\exp\overline{\mathfrak{a}}^+)K$, where K is a maximal compact subgroup of G and $\overline{\mathfrak{a}}^+$ a closed Weyl chamber in a maximal split torus $\mathfrak{a}$ of $\mathfrak{g}$. We denote by $\mu : G \to \overline{\mathfrak{a}}^+$ the corresponding *Cartan projection*; it is a continuous, surjective, *proper* map (this will be used all the time).

Let Σ be the set of restricted roots of $\mathfrak{a}$ in G, let $\Delta \subset \Sigma$ be the system of simple restricted roots associated with our choice of $\overline{\mathfrak{a}}^+$, and Σ^+ the corresponding set of positive restricted roots. To any subset $\theta \subset \Delta$ of the simple roots is associated a standard parabolic subgroup P_θ of G, with the convention that $\theta = \Delta$ corresponds to a minimal parabolic subgroup:

$$\mathrm{Lie}(P_\theta) = \mathfrak{z}_{\mathfrak{g}}(\mathfrak{a}) \oplus \bigoplus_{\alpha\in\Sigma^+} \mathfrak{g}_\alpha \oplus \bigoplus_{\alpha\in\Sigma^+\cap\mathrm{span}(\Delta\setminus\theta)} \mathfrak{g}_{-\alpha}.$$

These parameterize all conjugacy classes of parabolic subgroups of G. A Levi factor of P_θ is the closed subgroup L_θ with Lie algebra

$$\mathrm{Lie}(L_\theta) = \mathfrak{z}_{\mathfrak{g}}(\mathfrak{a}) \oplus \bigoplus_{\alpha\in\Sigma^+\cap\mathrm{span}(\Delta\setminus\theta)} (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}),$$

which is the intersection of P_θ with the opposite parabolic subgroup P_θ^- given by

$$\mathrm{Lie}(P_\theta^-) = \mathfrak{z}_{\mathfrak{g}}(\mathfrak{a}) \oplus \bigoplus_{\alpha\in\Sigma^+} \mathfrak{g}_{-\alpha} \oplus \bigoplus_{\alpha\in\Sigma^+\cap\mathrm{span}(\Delta\setminus\theta)} \mathfrak{g}_{\alpha}.$$

Example 2.1. Let $G = \mathrm{Sp}(4,\mathbf{R})$, stabilizer of the symplectic form $x_1\wedge x_3 + x_2\wedge x_4$ on $\mathbf{R}^4$. Then $\mathfrak{g}$ is the set of block matrices $\begin{pmatrix} B_1 & B_2 \\ B_3 & -{}^tB_1\end{pmatrix}$ with $B_i \in \mathrm{M}_2(\mathbf{R})$ and B_2, B_3 symmetric. We may take K to be $G\cap\mathrm{SO}(4) \simeq \mathrm{U}(2)$, and $\mathfrak{a}$ to be the set of diagonal matrices in $\mathfrak{g}$ with positive entries. Then

$$\Sigma = \{\pm\varepsilon_1 \pm \varepsilon_2, \pm 2\varepsilon_1, \pm 2\varepsilon_2\}.$$

We may also take $\overline{\mathfrak{a}}^+ = \{\mathrm{diag}(a_1, a_2, -a_1, -a_2) \mid a_1 \geq a_2 \geq 1\}$, so that $\Delta = \{\varepsilon_1 - \varepsilon_2, 2\varepsilon_2\}$. The standard parabolic subgroups of G are G itself, the proper maxi-

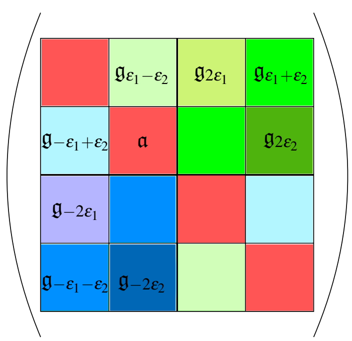

Fig. 1 The decomposition $\mathfrak{g} = \mathfrak{z}_{\mathfrak{g}}(\mathfrak{a}) \oplus \bigoplus_{\alpha\in\Sigma} \mathfrak{g}_\alpha$ for $G = \mathrm{Sp}(4,\mathbf{R})$

mal parabolic subgroups $P_{\{\varepsilon_1-\varepsilon_2\}}$ (stabilizer of the line $\mathbf{R}e_1$) and $P_{\{2\varepsilon_2\}}$ (stabilizer of the Lagrangian $\mathbf{R}e_1 \oplus \mathbf{R}e_2$), as well as the minimal parabolic subgroup P_Δ (stabilizer of the flag $(\mathbf{R}e_1 \subset \mathbf{R}e_1 \oplus \mathbf{R}e_2)$).

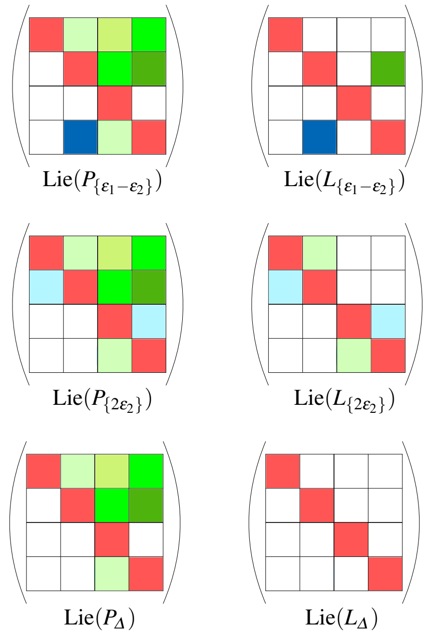

Fig. 2 The various proper parabolic subgroups P_θ and their Levi factors L_θ for $G = \mathrm{Sp}(4,\mathbf{R})$

For any $g \in G$ we can write $g = k_g(\exp\mu(g))k'_g$ where $k_g, k'_g \in K$ and $\mu(g) \in \overline{\mathfrak{a}}^+$, and $\langle \varepsilon_i, \mu(g)\rangle \geq 0$ is the logarithm of the i-th singular value of g (i.e. the logarithm of the length of the i-th principal axis of the ellipsoid $g \cdot \mathbb{B} \subset \mathbf{R}^4$, where $\mathbb{B}$ is the unit ball centered at 0 in $\mathbf{R}^4$).

For simplicity, we shall always assume θ to be stable under the opposition involution; this is the case for any θ when $G = \mathrm{Sp}(4,\mathbf{R})$.

Definition 2.1. Two points $x = gP_\theta \in G/P_\theta$ and $x' = g'P_\theta \in G/P_\theta$ are *transverse* if the corresponding parabolic subgroups $gP_\theta g^{-1}$ and $g'P_\theta g'^{-1}$ of G are opposite (i.e. their intersection is a reductive Lie group).

Example 2.2. Let $G = \mathrm{Sp}(4,\mathbf{R})$. For $\theta = \{\varepsilon_1 - \varepsilon_2\}$, the flag variety G/P_θ is the projective space $\mathbf{P}(\mathbf{R}^4)$; two lines ℓ, ℓ' of $\mathbf{R}^4$ are transverse if and only if $\ell \oplus \ell'^\perp = \mathbf{R}^4$. For $\theta = \{2\varepsilon_2\}$, the flag variety G/P_θ is the space of Lagrangians of $\mathbf{R}^4$; two Lagrangians V, V' are transverse if and only if $V \oplus V'^\perp = \mathbf{R}^4$. For $\theta = \Delta$, the flag variety G/P_θ is the space of maximal isotropic flags of $\mathbf{R}^4$; two such flags $(\ell \subset V)$ and $(\ell' \subset V')$ are transverse if and only if $\ell \oplus \ell'^\perp = V \oplus V'^\perp = \mathbf{R}^4$.

3 Discrete subgroups of Lie groups and their limit sets

We now fix a nonempty subset $\theta \subset \Delta$ (stable under the opposition involution). The *limit set* in G/P_θ of a Zariski-dense discrete subgroup Γ of G was introduced and studied by Guivarc'h [11] for $G = \mathrm{SL}_n(\mathbf{R})$ and by Benoist [4] in general. Here, following [8], we give a generalization of this notion to discrete subgroups of G that are not necessarily Zariski-dense.

Let $x_\theta = eP_\theta \in G/P_\theta$ be the basepoint of G/P_θ. We define a map $\Xi_\theta : G \to G/P_\theta$ as follows: for any $g \in G$, we choose $k_g, k'_g \in K$ such that $g = k_g(\exp \mu(g))k'_g$, and set

$$\Xi_\theta(g) = k_g \cdot x_\theta \in G/P_\theta. \tag{3.1}$$

This does not depend on the choice of k_g, k'_g as soon as $\langle \alpha, \mu(g) \rangle > 0$ for all $\alpha \in \theta$.

Example 3.1. Let $G = \mathrm{Sp}(4,\mathbf{R})$ and $K, \overline{\mathfrak{a}}^+$ be as in Example 2.1.

For $\theta = \{\varepsilon_1 - \varepsilon_2\}$, we have $G/P_\theta = \mathbf{P}(\mathbf{R}^4)$ and $\Xi_\theta(g)$ is the direction of the principal axis of the ellipsoid $g \cdot \mathbb{B} \subset \mathbf{R}^4$.

For $\theta = \{2\varepsilon_2\}$, the space G/P_θ is the space of Lagrangians of $\mathbf{R}^4$ and $\Xi_\theta(g)$ is the Lagrangian spanned by the two first principal axes of the ellipsoid $g \cdot \mathbb{B} \subset \mathbf{R}^4$.

If g is proximal in G/P_θ (in the sense that there exists $x_g^+, x_g^- \in G/P_\theta$ such that $g^n \cdot x \to x_g^+$ for all $x \in G/P_\theta$ which are not transverse to x_g^-), then the sequence $(\Xi_\theta(g^n))_{n\in\mathbf{N}}$ converges to the attracting fixed point x_g^+ of g in G/P_θ (Lemma 6.2).

Definition 3.1. Let Γ be a discrete subgroup of G such that $\langle \alpha, \mu(\Gamma) \rangle \subset \mathbf{R}_+$ is unbounded for all $\alpha \in \theta$. The *limit set* $\Lambda_\Gamma^{G/P_\theta}$ of Γ in G/P_θ is the set of all possible limits of sequences $(\Xi_\theta(\gamma_n))_{n\in\mathbf{N}}$ for $(\gamma_n) \in \Gamma^{\mathbf{N}}$ with $\langle \alpha, \mu(\gamma_n) \rangle \to +\infty$ for all $\alpha \in \theta$.

The limit set $\Lambda_\Gamma^{G/P_\theta}$ is Γ-invariant and does not depend on the choice of Cartan decomposition (see Section 6.1). If Γ is Zariski-dense in G, then it is well known that it contains an element which is proximal in G/P_θ (in particular $\langle \alpha, \mu(\Gamma) \rangle \subset \mathbf{R}_+$ is unbounded for all $\alpha \in \theta$), and Benoist proved that in this case $\Lambda_\Gamma^{G/P_\theta}$ is the closure in G/P_θ of the set of attracting fixed points of these elements.

Definition 3.2. Let Γ be a discrete group and $\rho : \Gamma \to G/P_\theta$ a representation. We say that ρ is *P_θ-divergent* if $\langle \alpha, \mu(\rho(\gamma)) \rangle \xrightarrow[\gamma\to\infty]{} +\infty$ for all $\alpha \in \theta$.

In this case ρ has finite kernel and discrete image, and the limit set $\Lambda_{\rho(\Gamma)}^{G/P_\theta}$ is well defined. When G has real rank one, P_θ-divergence is *equivalent* to ρ having finite kernel and discrete image; in this case, $\Lambda_{\rho(\Gamma)}^{G/P_\theta}$ is the usual limit set of $\rho(\Gamma)$ in $G/P_\theta \simeq \partial_\infty(G/K)$.

4 Existence and point-by-point construction of continuous equivariant boundary maps

In certain situations, not only does $\rho(\Gamma)$ have a limit set in G/P_θ, but there is in fact a continuous ρ-equivariant boundary map from some boundary $\partial_\infty\Gamma$ of Γ to G/P_θ, identifying $\partial_\infty\Gamma$ with the limit set of $\rho(\Gamma)$. This is the case for P_θ-Anosov representations ρ, but also for more general representations. We give a point-by-point construction of such a boundary map.

4.1 The main result on boundary maps

For a word hyperbolic group Γ and a P_θ-regular representation $\rho : \Gamma \to G$, the existence and nice properties of a ρ-equivariant boundary map $\xi : \partial_\infty\Gamma \to G/P_\theta$ depend on "how fast" $\langle \alpha, \mu(\rho(\gamma))\rangle \underset{\gamma\to\infty}{\longrightarrow} +\infty$ for $\alpha \in \theta$, as follows:

Theorem 4.1 ([8, Th. 1.1]). *Let Γ be a word hyperbolic group and $\rho : \Gamma \to G$ a representation.*

1. *If there exists $C > 0$ such that $\langle \alpha, \mu(\rho(\gamma_n))\rangle \geq 2\log|\gamma|_\Gamma - C$ for all $\alpha \in \theta$ and $\gamma \in \Gamma$, then the map $\Xi := \Xi_\theta \circ \rho : \Gamma \to G/P_\theta$ extends continuously to a map*

   $$\Xi \sqcup \xi : \Gamma \cup \partial_\infty\Gamma \to G/P_\theta.$$

   The continuous boundary map $\xi : \partial_\infty\Gamma \to G/P_\theta$ is ρ-equivariant and independent of any choice, and $\xi(\partial_\infty\Gamma) = \Lambda^{G/P_\theta}_{\rho(\Gamma)}$.
2. *If moreover for any $\alpha \in \theta$ and any $\gamma \in \Gamma$ of infinite order,*

   $$\langle \alpha, \mu(\rho(\gamma^n))\rangle - 2\log n \underset{n\to+\infty}{\longrightarrow} +\infty,$$

   then ξ is dynamics-preserving for ρ.
3. *If moreover for any $\alpha \in \Sigma^+_\theta$ and any geodesic ray $(\gamma_n)_{n\in\mathbf{N}}$ in the Cayley graph of Γ the map $n \mapsto \langle \alpha, \mu(\rho(\gamma_n))\rangle$ is a quasi-isometric embedding of $\mathbf{N}$ into $\mathbf{R}_+$, then ξ is transverse; moreover, ρ is P_θ-Anosov and ξ defines a homeomorphism $\partial_\infty\Gamma \simeq \Lambda^{G/P_\theta}_{\rho(\Gamma)}$.*

In (1), we denote by $|\gamma|_\Gamma$ the word length of γ (with respect to some fixed finite generating set). In (2), by "dynamics-preserving" we mean that the image by ξ of the attracting fixed point of $\gamma \in \Gamma$ in $\partial_\infty\Gamma$ is an attracting fixed point of $\rho(\gamma)$ in G/P_θ. In (3), by "transverse" we mean that $\xi(\eta), \xi(\eta') \in G/P_\theta$ are transverse in the sense of Definition 2.1 for any $\eta \neq \eta'$ in $\partial_\infty\Gamma$; we refer to Section 5.1 for the definition of P_θ-Anosov. We denote by Σ^+_θ be the set of positive roots that do *not* belong to the span of $\Delta \smallsetminus \theta$.

Example 4.1. Let $G = \mathrm{Sp}(4,\mathbf{R})$.
For $\theta = \{\varepsilon_1 - \varepsilon_2\}$ we have $\Sigma_\theta^+ = \{\varepsilon_1 - \varepsilon_2, \varepsilon_1 + \varepsilon_2, 2\varepsilon_1\}$.
For $\theta = \{2\varepsilon_2\}$ we have $\Sigma_\theta^+ = \{\varepsilon_1 + \varepsilon_2, 2\varepsilon_1, 2\varepsilon_2\}$.
For $\theta = \Delta$ we have $\Sigma_\theta^+ = \Sigma^+$.

Remark 4.1. Theorem 4.1.(1) extends a rank-one result of Floyd [6] to the setting of higher real rank.

Here is a case where the assumption of (1) is satisfied but the conclusions of (2) and (3) fail:

Example 4.2. Let $G = \mathrm{SL}_2(\mathbf{R})$, with Cartan projection $\mu : G \to \mathbf{R}_+$ obtained by identifying $\overline{\mathfrak{a}}^+$ with $\mathbf{R}_+$. Let Γ be a finitely generated Schottky subgroup of G containing a parabolic element u. There is a constant $C > 0$ such that $\mu(\rho(\gamma)) \geq 2\log|\gamma|_\Gamma - C$ for all $\gamma \in \Gamma$. This growth rate cannot be improved since $\mu(u^n) = 2\log n + O(1)$. The continuous, equivariant boundary map $\xi : \partial_\infty\Gamma \to \partial_\infty\mathbb{H}^2$ given by Theorem 4.1.(1) is not dynamics-preserving since the fixed point of u is neither attracting nor repelling in $G/P_\theta = \partial_\infty\mathbb{H}^2$. The map ξ is also not transverse (i.e. not injective) since $\xi(\lim_{+\infty} u^n) = \xi(\lim_{+\infty} u^{-n})$.

We refer to [8, Ex. A.5] for an example where the assumptions of (1) and (3) are both satisfied but the conclusion of (2) fails.

It is likely that Theorem 4.1 could be extended to nonhyperbolic groups Γ (for some suitable notion of boundary $\partial_\infty\Gamma$), providing various generalizations of Anosov representations.

4.2 A refinement of Theorem 4.1.(1)

Here is a slightly stronger version of Theorem 4.1.(1).

Theorem 4.2 ([8, Th. 5.3]). *Let Γ be a word hyperbolic group and $\rho : \Gamma \to G$ a representation.*

1a. Suppose ρ has the gap summation property *with respect to θ, i.e. for any $\alpha \in \theta$ and any geodesic ray $(\gamma_n)_{n\in\mathbf{N}}$ in the Cayley graph of Γ,*

$$\sum_{n\in\mathbf{N}} e^{-\langle \alpha, \mu(\rho(\gamma_n))\rangle} < +\infty. \tag{4.1}$$

(This is the case for instance if for any $\alpha \in \theta$ and any geodesic ray $(\gamma_n)_{n\in\mathbf{N}}$ we have $\langle \alpha, \mu(\rho(\gamma_n))\rangle \geq 2\log n - O(1)$.)
Then the map $\Xi := \Xi_\theta \circ \rho : \Gamma \to G/P_\theta$ induces *a ρ-equivariant boundary map*

$$\xi : \partial_\infty\Gamma \to G/P_\theta$$

(i.e. $\xi(\eta) = \lim_n \Xi(\gamma_n)$ for any quasi-geodesic ray $(\gamma_n)_{n\in\mathbf{N}}$ in the Cayley graph of Γ with endpoint $\eta \in \partial_\infty\Gamma$). The map ξ is independent of any choice.

1b. Suppose ρ *has the* uniform gap summation property *with respect to* θ*, i.e. the series* (4.1) *converges uniformly for all geodesic rays* $(\gamma_n)_{n\in\mathbf{N}}$ *with* $\gamma_0 = e$*. (This is the case for instance if there exists* $C > 0$ *such that* $\langle \alpha, \mu(\rho(\gamma_n))\rangle \geq 2\log n - C$ *for any* $\alpha \in \theta$*, any geodesic ray* $(\gamma_n)_{n\in\mathbf{N}}$ *with* $\gamma_0 = e$*, and any* $n \in \mathbf{N}$*.)*
Then $\Xi \sqcup \xi : \Gamma \cup \partial_\infty \Gamma \to G/P_\theta$ *is continuous and* $\xi(\partial_\infty \Gamma) = \Lambda^{G/P_\theta}_{\rho(\Gamma)}$.

5 Anosov representations

5.1 The original definition of Anosov representations

The original definition of an Anosov representation uses the *flow space* $\mathscr{G}_\Gamma$ of Γ: this notion is not totally straightforward in general, but if $\Gamma = \pi_1(M)$ for some closed, negatively curved, Riemannian manifold, then $\mathscr{G}_\Gamma$ is just $T^1(\tilde{M})$ where $\tilde{M}$ is a universal covering of M. More precisely, let L_θ be a Levi factor of P_θ as in Section 2. The definition of a P_θ-Anosov representation uses the G/L_θ-bundle

$$\mathscr{E}(\rho) := \Gamma \backslash (\mathscr{G}_\Gamma \times G/L_\theta)$$

over $\Gamma\backslash\mathscr{G}_\Gamma$ and the "vertical tangent bundle"

$$T^{\mathrm{v}}\mathscr{E}(\rho) := \Gamma \backslash (\mathscr{G}_\Gamma \times T(G/L_\theta))$$

over $\mathscr{E}(\rho)$ (where Γ acts diagonally, and via ρ on the second factor). Note that G/L_θ identifies with the unique open G-orbit in $G/P_\theta \times G/P_\theta^-$; this determines subbundles E^+, E^- of $T(G/L_\theta)$ corresponding to the two factors, which themselves induce subbundles (still denoted E^+, E^-) of $T^{\mathrm{v}}\mathscr{E}(\rho)$. The geodesic flow $\{\varphi_t\}_{t\in\mathbf{R}}$ naturally acts on the product $\mathscr{G}_\Gamma \times T(G/L_\theta)$ (leaving the second coordinate unchanged), hence on its quotient $T^{\mathrm{v}}\mathscr{E}(\rho)$; the subbundles $E^\pm$ are flow-invariant.

Definition 5.1. The representation $\rho : \Gamma \to G$ is P_θ*-Anosov* if there is a continuous section $\sigma : \Gamma\backslash\mathscr{G}_\Gamma \to \mathscr{E}(\rho)$ with the following properties:

1. σ is flow-equivariant (i.e. its image $F := \sigma(\Gamma\backslash\mathscr{G}_\Gamma)$ is flow-invariant);
2. the action of the flow on the vector bundle $E^+|_F \subset T^{\mathrm{v}}\mathscr{E}(\rho)$ is dilating;
3. the action of the flow on the vector bundle $E^-|_F \subset T^{\mathrm{v}}\mathscr{E}(\rho)$ is contracting.

The section σ satisfying (1) can be seen as a flow-invariant, ρ-equivariant, continuous map $\sigma' : \mathscr{G}_\Gamma \to G/L_\theta$. The existence of such a map σ' is equivalent to the existence of a ρ-equivariant boundary map $\xi : \partial_\infty\Gamma \to G/P_\theta$ given by

$$\sigma'(v) = \lim_{t\to+\infty} (\xi(\varphi_t \cdot v), \xi(\varphi_{-t} \cdot v))$$

for all $v \in \mathscr{G}_\Gamma$, such that ξ is continuous, dynamics-preserving, and transverse. Thus $\rho : \Gamma \to G$ is P_θ-Anosov if and only if there exists a continuous, dynamics-preserving, transverse, ρ-equivariant boundary map $\xi : \partial_\infty \Gamma \to G/P_\theta$ with some good dilation and contraction properties as in Definition 5.1.(2)–(3).

5.2 Characterizations in terms of the Cartan projection

We give several characterizations of Anosov representations. Some of them replace the contraction/dilation properties of Definition 5.1.(2)–(3) with growth properties of the Cartan projection (Theorem 5.1) or Lyapunov projection (Theorem 5.2). One of them (Theorem 5.1.(5)) shows that the sufficient condition given by Theorem 4.1 is actually necessary.

Theorem 5.1 ([8, Th. 1.3 & 4.1]). *Let Γ be a word hyperbolic group and $\rho : \Gamma \to G$ a representation. Suppose there exists a continuous, ρ-equivariant, transverse map $\xi : \partial_\infty \Gamma \to G/P_\theta$. Then the following conditions are equivalent:*

1. *ρ is P_θ-Anosov;*
2. *There exists a continuous, ρ-equivariant, dynamics-preserving, transverse map $\xi : \partial_\infty \Gamma \to G/P_\theta$ and for any $\alpha \in \theta$,*

$$\langle \alpha, \mu(\rho(\gamma)) \rangle \underset{\gamma\to\infty}{\longrightarrow} +\infty;$$

3. *There exist a continuous, ρ-equivariant, dynamics-preserving, transverse map $\xi : \partial_\infty \Gamma \to G/P_\theta$ and constants $c, C > 0$ such that for any $\alpha \in \theta$ and $\gamma \in \Gamma$,*

$$\langle \alpha, \mu(\rho(\gamma)) \rangle \geq c\,\|\mu(\rho(\gamma))\| - C;$$

4. *There exist a continuous, ρ-equivariant, dynamics-preserving, transverse map $\xi : \partial_\infty \Gamma \to G/P_\theta$ and constants $c, C > 0$ such that for any $\alpha \in \theta$ and $\gamma \in \Gamma$,*

$$\langle \alpha, \mu(\rho(\gamma)) \rangle \geq c\,|\gamma|_\Gamma - C;$$

5. *There exist $\kappa, \kappa' > 0$ such that for any $\alpha \in \Sigma_\theta^+$ and any geodesic ray $(\gamma_n)_{n\in\mathbf{N}}$ with $\gamma_0 = e$ in the Cayley graph of Γ, the map $n \mapsto \langle \alpha, \mu(\rho(\gamma_n)) \rangle$ is a (κ, κ')-quasi-isometric embedding of $\mathbf{N}$ into $\mathbf{R}_+$.*

Remark 5.1. The implications $(1) \Leftrightarrow (2) \Leftrightarrow (3) \Leftrightarrow (4) \Leftarrow (5)$ of Theorem 5.1 (which contain Theorem 4.1.(3)) also follow from the independent work of Kapovich–Leeb–Porti.

Remark 5.2. When $\rho(\Gamma)$ is Zariski-dense in G, the existence of a continuous, ρ-equivariant, dynamics-preserving, transverse map $\xi : \partial_\infty \Gamma \to G/P_\theta$ is sufficient for ρ to be P_θ-Anosov [10, Th. 4.11]. However, this is *not* true in general, even when ρ is semisimple (i.e. the Zariski closure of $\rho(\Gamma)$ is reductive): see [8, Ex. 7.15].

5.3 Characterizations in terms of the Lyapunov projection

We also prove the following characterizations in terms of the Lyapunov projection $\lambda : G \to \overline{\mathfrak{a}}^+$, which is the projection associated with the Jordan decomposition.

Theorem 5.2 ([8, Th. 1.7 & 4.2]). *Let Γ be a word hyperbolic group and $\rho : \Gamma \to G$ a representation. Suppose there exists a continuous, ρ-equivariant, transverse map $\xi : \partial_\infty \Gamma \to G/P_\theta$. Then the following conditions are equivalent:*

1. *ρ is P_θ-Anosov;*
2. *There exists a continuous, ρ-equivariant, dynamics-preserving, transverse map $\xi : \partial_\infty \Gamma \to G/P_\theta$ and for any $\alpha \in \theta$,*
$$\langle \alpha, \lambda(\rho(\gamma)) \rangle \underset{\gamma \to \infty}{\longrightarrow} +\infty;$$
3. *There exist a continuous, ρ-equivariant, dynamics-preserving, transverse map $\xi : \partial_\infty \Gamma \to G/P_\theta$ and a constant $c > 0$ such that for any $\alpha \in \theta$ and $\gamma \in \Gamma$,*
$$\langle \alpha, \lambda(\rho(\gamma)) \rangle \geq c \, \| \lambda(\rho(\gamma)) \|;$$
4. *There exist a continuous, ρ-equivariant, dynamics-preserving, transverse map $\xi : \partial_\infty \Gamma \to G/P_\theta$ and a constant $c > 0$ such that for any $\alpha \in \theta$ and $\gamma \in \Gamma$,*
$$\big\langle \alpha, \, \lambda(\rho(\gamma)) \big\rangle \geq c \, |\gamma|_\infty.$$

In (4), we denote by $|\gamma|_\infty := \lim_n |\gamma^n|_\Gamma / n$ the stable length of γ.

Remark 5.3. Let $\ell_\Gamma(\gamma) := \inf_{\beta \in \Gamma} |\beta \gamma \beta^{-1}|_\Gamma$ be the translation length of γ on the Cayley graph of Γ. It is well known that $\ell_\Gamma(\gamma) - 16\delta \leq |\gamma|_\infty \leq \ell_\Gamma(\gamma)$ where δ is the hyperbolicity constant of Γ. Therefore, (4) is also equivalent to:

5. There exist a continuous, ρ-equivariant, dynamics-preserving, transverse map $\xi : \partial_\infty \Gamma \to G/P_\theta$ and constants $c, C > 0$ such that for any $\alpha \in \theta$ and $\gamma \in \Gamma$,
$$\big\langle \alpha, \, \lambda(\rho(\gamma)) \big\rangle \geq c \, \ell_\Gamma(\gamma) - C.$$

5.4 Anosov representations and character varieties

A P_θ-Anosov representation $\rho : \Gamma \to G$ is not necessarily semisimple, even when P_θ is a minimal parabolic subgroup of G. In the course of the proof of Theorem 5.2 we establish the following statement, of independent interest.

Proposition 5.1 ([8, Prop. 1.8]). *Let Γ be a word hyperbolic group, G a real reductive Lie group, and $\theta \subset \Delta$ a nonempty subset of the simple restricted roots of G. Let $\rho : \Gamma \to G$ be a representation and ρ^{ss} its semisimplification. Then*
$$\rho \text{ is } P_\theta\text{-Anosov} \iff \rho^{ss} \text{ is } P_\theta\text{-Anosov.}$$

Since the character variety of Γ in G can be viewed as the quotient of $\mathrm{Hom}(\Gamma, G)$ by the relation "having the same semisimplification", Proposition 5.1 means that the notion of being P_θ-Anosov is well defined in the character variety.

6 Construction of boundary maps: proofs

To make things more concrete, we take $G = \mathrm{Sp}(4, \mathbf{R})$ and $\theta = \{\varepsilon_1 - \varepsilon_2\}$, so that G/P_θ identifies with $\mathbf{P}(\mathbf{R}^4)$ and $\Xi_\theta(g)$ is the direction of the principal axis of the ellipsoid $g \cdot \mathbb{B} \subset \mathbf{R}^4$ (see Examples 2.1, 2.2, 3.1). We set $x_0 := [e_1] \in \mathbf{P}(\mathbf{R}^4) = G/P_\theta$ and $y_0 := [e_3] \in \mathbf{P}(\mathbf{R}^4) = G/P_\theta$. We endow $G/P_\theta = \mathbf{P}(\mathbf{R}^4)$ with the K-invariant metric

$$d([v],[v']) = |\sin \measuredangle(v, v')|.$$

6.1 Easier part

Our proof of Theorem 4.1.(1) relies on the following elementary estimates.

Lemma 6.1. *For any compact subset M of G there exists $C > 0$ such that for any $g \in G$ and $m \in M$,*

1. $d(\Xi_\theta(gm), \Xi_\theta(g)) \leq Ce^{-\langle \varepsilon_1 - \varepsilon_2, \mu(g) \rangle}$;
2. $d(\Xi_\theta(mg), m \cdot \Xi_\theta(g)) \leq Ce^{-\langle \varepsilon_1 - \varepsilon_2, \mu(g) \rangle}$.

Consequences of Lemma 6.1:

- $\lim_n \Xi_\theta(m \cdot g_n) = m \cdot \lim_n \Xi_\theta(g_n)$, and so $\Lambda^{\mathbf{P}(\mathbf{R}^4)}_{\rho(\Gamma)}$ is $\rho(\Gamma)$-invariant and ξ (if well defined) is ρ-equivariant;
- $d(mk_{m^{-1}gm} \cdot x_0, k_g \cdot x_0) \to 0$ as $\langle \varepsilon_1 - \varepsilon_2, \mu(g) \rangle \to +\infty$, as so $\Lambda^{\mathbf{P}(\mathbf{R}^4)}_{\rho(\Gamma)}$ and ξ (if well defined) do not depend on the choice of Cartan decomposition of G (whereas Ξ_θ of course depends on it);
- ξ is well defined as soon as the gap summation property of Theorem 4.2 is satisfied;
- $\Xi \sqcup \xi : \Gamma \cup \partial_\infty \Gamma \to \mathbf{P}(\mathbf{R}^4)$ is continuous as soon as the *uniform* gap summation property of Theorem 4.2 is satisfied.

Note that $\xi(\partial_\infty \Gamma)$ is always contained in the limit set $\Lambda^{\mathbf{P}(\mathbf{R}^4)}_{\rho(\Gamma)}$ by construction; there is equality when $\Xi \sqcup \xi$ is continuous (using the properness of μ and the compactness of $\partial_\infty \Gamma$). This completes the proof of Theorem 4.1.(1).

Theorem 4.1.(2) is based on the following observation, whose proof uses similar arguments to that of Lemma 6.1.

Lemma 6.2. *For any $g \in G$ which is proximal in $G/P_\theta = \mathbf{P}(\mathbf{R}^4)$,*

$$\Xi_\theta(g^n) \underset{n\to+\infty}{\longrightarrow} \xi_g^+,$$

where ξ_g^+ is the attracting fixed point of g in G/P_θ.

6.2 Harder part

Theorem 4.1.(3) is more difficult. Here is the main technical lemma.

Lemma 6.3. *Let $(\gamma_n) \in \Gamma^{\mathbf{N}}$ be a geodesic ray with endpoint $\eta \in \partial_\infty\Gamma$. Write $\rho(\gamma_n) = k_n a_n k_n' \in K(\exp \overline{\mathfrak{a}}^+)K$. If the map $n \mapsto \langle \varepsilon_1 - \varepsilon_2, \mu(\rho(\gamma_n))\rangle$ is a quasi-isometric embedding of $\mathbf{N}$ into $\mathbf{R}_+$, then there exists $0 < \delta \leq 1$ such that for all large enough $n \in \mathbf{N}$,*

$$\limsup_{m\to+\infty} d\big((a_n^{-1}k_n^{-1}k_{n+m}a_{n+m})\cdot x_0, x_0\big) \leq 1-\delta.$$

In particular, for all large enough n,

$$d\big(a_n^{-1}k_n^{-1}\cdot\xi(\eta), x_0\big) \leq 1-\delta.$$

Note that, by definition of ξ, the sequence $(k_n^{-1}\cdot\xi(\eta))_{n\in\mathbf{N}}$ converges to x_0, which is the repelling fixed point of all elements a_n^{-1}. Lemma 6.3 states that this convergence is so fast, compared to the growth of a_n^{-1}, that $a_n^{-1}k_n^{-1}\cdot\xi(\eta)$ does not go too far away from x_0.

Proof (Proof of Theorem 4.1.(3) *using Theorem 5.1 and Lemma 6.3).* It is sufficient to prove the transversality of ξ. Indeed, the fact that ρ is P_θ-Anosov then follows from Theorem 5.1, and the fact that ξ defines a homeomorphism onto its image follows from the continuity and injectivity of ξ and the compactness of $\partial_\infty\Gamma$.

Consider $\eta \in \partial_\infty\Gamma$. Let us prove that $\xi(\eta) \notin \xi(\eta')^\perp$ for all $\eta' \in \partial_\infty\Gamma \smallsetminus \{\eta\}$. Let $0 < \delta \leq 1$ be given by Lemma 6.3. We shall prove that for any $\eta' \in \partial_\infty\Gamma \smallsetminus \{\eta\}$ we can find (infinitely many) $n \in \mathbf{N}$ such that

$$d\big(\rho(\gamma_n)^{-1}\cdot\xi(\eta), \rho(\gamma_n)^{-1}\cdot\xi(\eta')^\perp\big) \geq \frac{\delta}{2}. \tag{6.1}$$

In particular, the point $\xi(\eta)$ does not belong to the hyperplane $\xi(\eta')^\perp$.

(i) We can find a subsequence $(\gamma_{\phi(n)})_{n\in\mathbf{N}}$ such that $(\gamma_{\phi(n)}^{-1})_{n\in\mathbf{N}}$ converges to some point $\eta'' \in \partial_\infty\Gamma$ and that $\lim \gamma_{\phi(n)}^{-1}\cdot\eta' = \eta''$ for all $\eta' \in \partial_\infty\Gamma \smallsetminus \{\eta\}$. By ρ-equivariance and continuity of ξ, for all $\eta' \in \partial_\infty\Gamma \smallsetminus \{\eta\}$,

$$\lim_n \rho(\gamma_{\phi(n)})^{-1}\cdot\xi(\eta') = \xi\big(\lim_n \gamma_{\phi(n)}^{-1}\cdot\eta'\big) = \xi(\eta'').$$

(ii) On the other hand,

$$\xi(\eta'') = \lim_{n\to+\infty} k'^{-1}_{\phi(n)}\cdot y_0,$$

because

$$\rho(\gamma_n)^{-1} = (k_n'^{-1}w)\,(w^{-1}a_n^{-1}w)\,(w^{-1}k_n^{-1}) \in K(\exp\overline{\mathfrak{a}}^+)K$$

and $y_0 = w \cdot x_0$ (where w is the permutation matrix switching e_1 and e_3, and e_2 and e_4), and $\Xi \sqcup \xi$ is continuous. Thus

$$\lim_n \rho(\gamma_{\phi(n)})^{-1} \cdot \xi(\eta') = \lim_{n\to+\infty} k_{\phi(n)}'^{-1} \cdot y_0.$$

(iii) By Lemma 6.3, there exists $0 < \delta \leq 1$ such that for all large enough n,

$$d\big(\rho(\gamma_n)^{-1} \cdot \xi(\eta), k_n'^{-1} \cdot x_0\big) \leq 1 - \delta.$$

By definition of d, for any $x \in \mathbf{P}(\mathbf{R}^4)$,

$$d(x, y_0^{\perp})^2 + d(x, x_0)^2 = 1,$$

and so for all large enough n,

$$d\big(\rho(\gamma_n)^{-1} \cdot \xi(\eta), k_n'^{-1} \cdot y_0^{\perp}\big) \geq \sqrt{1 - (1-\delta)^2} \geq \delta.$$

(iv) Putting together (ii) and (iii), we see that for any $\eta' \in \partial_\infty\Gamma \smallsetminus \{\eta\}$ we have, for all large enough n,

$$d\big(g_{\phi(n)}^{-1} \cdot \xi(\eta), g_{\phi(n)}^{-1} \cdot \xi(\eta')^{\perp}\big) \geq \frac{\delta}{2},$$

proving (6.1). ☐

Remark 6.1. Lemma 6.3 states an expansion property for the action of $(\rho(\gamma_n^{-1}))_{n\in\mathbf{N}}$ on $\mathbf{P}(\mathbf{R}^4)$ at $\xi(\eta)$. Indeed, for any large enough $n \in \mathbf{N}$, the open set

$$\mathscr{U}_n := \left\{ x \in \mathbf{P}(\mathbf{R}^4) \;\middle|\; d(a_n^{-1}k_n^{-1} \cdot x, x_0) < 1 - \frac{\delta}{2} \right\}$$

is a neighborhood of $\xi^+(\eta)$ by Lemma 6.3. There is a constant $C > 0$ such that for any n the element a_n is $Ce^{-\langle \varepsilon_1 - \varepsilon_2, \mu(\rho(\gamma_n))\rangle}$-Lipschitz on $B_{x_0}(1 - \frac{\delta}{2})$. Therefore $\rho(\gamma_n^{-1})$ is $C^{-1}e^{\langle \varepsilon_1 - \varepsilon_2, \mu(\rho(\gamma_n))\rangle}$-expanding on $\mathscr{U}_n$, where $C^{-1}e^{\langle \varepsilon_1 - \varepsilon_2, \mu(\rho(\gamma_n))\rangle} \to +\infty$.

7 Applications of Theorems 5.1 and 5.2 to properly discontinuous actions on non-Riemannian homogeneous spaces

8 Domains of discontinuity and compactification of Riemannian locally symmetric spaces

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