Parabolic Homogeneous Spaces of $\mathrm{Sp}(4,\mathbb{R})$

Jean-Philippe Burelle

Abstract The purpose of these notes is to describe the homogeneous spaces of $\mathrm{Sp}(4,\mathbb{R})$ for which the stabilizer of a point is a parabolic subgroup. There are three conjugacy classes of parabolic subgroups in $\mathrm{Sp}(4,\mathbb{R})$: the stabilizer of an isotropic line, the stabilizer of a Lagrangian plane, and the stabilizer of an isotropic flag. The corresponding homogeneous spaces are respectively $\mathbb{RP}^3$ endowed with a contact structure (or *photon space*), the $2+1$-dimensional Einstein universe, and the space of pointed photons in the Einstein universe. We will try to focus on the interactions between the projective contact geometry of photon space and the flat conformal Lorentzian geometry of the Einstein universe.

1 Root Systems and Parabolic Subgroups

Parabolic subgroups are classified by subsets of a fixed set of simple roots. In this section we explicitly describe a set of positive roots and classify the parabolic subgroups of $\mathrm{Sp}(4,\mathbb{R})$.

Let $j = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $J = \begin{pmatrix} 0 & j \\ j & 0 \end{pmatrix}$ as a 4×4 block matrix. We will use the skew symmetric form ω given by the matrix J as our standard bilinear form throughout.

Using this bilinear form, the Lie algebra $\mathfrak{sp}(4,\mathbb{R})$ is

$$\left\{ \begin{pmatrix} A & B \\ C & -Adj(A) \end{pmatrix} \mid B,C \in \mathfrak{sl}(2,\mathbb{R}) \right\}$$

Jean-Philippe Burelle
University of Maryland, College Park, e-mail: jburelle@umd.edu

Where $Adj(A)$ denotes the adjugate matrix of A. The subalgebra consisting of diagonal matrices $\begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & -b & 0 \\ 0 & 0 & 0 & -a \end{pmatrix}$ is a Cartan subalgebra, and the corresponding roots are the covectors $\pm a \pm b, \pm 2a, \pm 2b$. The subset $2a, a-b, a+b, 2b$ is a set of positive roots. These corresspond to the root spaces:

$$\begin{pmatrix} 0 & 0 & 0 & x \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & y & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & y \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & z & 0 \\ 0 & 0 & 0 & -z \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & w & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

A set of simple roots is formed by $a-b, 2b$. Now we are ready to describe the parabolic subgroups.

Corresponding to the subset $\{a-b\}$, we get the Lie algebra corresponding to the subgroup:

$$\begin{pmatrix} A & 0 \\ B & C \end{pmatrix}$$

which is the stabilizer the Lagrangian plane where the first two coordinates are zero.

Corresponding to the subset $\{2b\}$, we get the Lie algebra corresponding to the subgroup:

$$\begin{pmatrix} a & 0 & 0 & 0 \\ & & & 0 \\ & B & & 0 \\ & d & & f \end{pmatrix}$$

which is the stabilizer of a the line spanned by the fourth coordinate vector.

Finally, the empty subset corresponds to the subgroup of lower triangular matrices, which is the stabilizer of an isotropic flag.

2 The $\mathrm{SO}(3,2)$ model

In this section we will describe the local isomorphism between the Lie groups $\mathrm{Sp}(4,\mathbb{R})$ and $\mathrm{SO}(3,2)$, as well as the correspondence between the parabolic homogeneous spaces.

We start with an abstract four-dimensional real vector space V, endowed with a symplectic form ω. Then, the group $G = \mathrm{Sp}(V,\omega)$ of linear automorphisms preserving ω is isomorphic to $\mathrm{Sp}(4,\mathbb{R})$.

The second exterior power $\Lambda^2 V$ is a 6-dimensional real vector space on which G acts linearly. We endow this exterior power with a natural bilinear form $(,)$ defined by:

$$(v \wedge w, v' \wedge w')\mathrm{vol} = v \wedge w \wedge v' \wedge w'$$

where vol is a fixed element of $\Lambda^4 V$.

Proposition 1. *This form is symmetric, nondegenerate, and has signature* $(3,3)$.

Proof. Pick a basis e_i for V, $1 \le i \le 4$. It induces a basis $e_i \wedge e_j$ $i<j$ of $\Lambda^2 V$. Choose $\mathrm{vol} = e_1 \wedge e_2 \wedge e_3 \wedge e_4$. *Then, in the induced basis the form* $(,)$ *has matrix*

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

This is the matrix of a nondegenerate signature (3,3) bilinear form.

This bilinear form is preserved by $\mathrm{SL}(4,\mathbb{R})$ since for a linear map $L : V \to V$ we have

$$Lv \wedge Lw \wedge Lv' \wedge Lw' = \det(L) v \wedge w \wedge v' \wedge w'.$$

This gives us a homomorphism $\mathrm{SL}(4,\mathbb{R}) \to \mathrm{SO}(3,3)$. This corresponds to the Lie algebra isomorphism $\mathfrak{sl}(4,\mathbb{R}) \to \mathfrak{so}(3,3)$ coming from the Dynkin diagrams $A_3 \sim D_3$.

We will show that under this homomorphism, the subgroup $\mathrm{Sp}(4,\mathbb{R})$ maps onto a subgroup isomorphic to $\mathrm{SO}(3,2)$.

$\mathrm{Sp}(V,\omega)$ preserves $\omega \in \Lambda^2 V^*$ by definition. We define a dual ω^* to the skew-symmetric form by:

$$\omega(v,w) = (\omega^*, v \wedge w).$$

This element is then also preserved by all elements of $\mathrm{Sp}(V,\omega)$. This implies that its orthogonal complement with respect to $(,)$ is also preserved. Note that

$$(\omega^*, \omega^*) = \omega^* \wedge \omega^* \mathrm{vol} \neq 0,$$

since ω is non-degenerate. This means that the orthogonal complement of ω has signature $(3,2)$ or $(2,3)$. The action of $\mathrm{Sp}(V,\omega)$ on $\omega^{*\perp}$ thus defines a map $\mathrm{Sp}(V,\omega) \to \mathrm{SO}(3,2)$.

2.1 Images of Parabolic Subgroups

Proposition 2. *Under this homomorphism, the stabilizer of a Lagrangian plane maps to the stabilizer of an isotropic line.*

Proof. Planes in V correspond to simple 2-vectors in $\Lambda^2 V$, up to scaling by a real number. A Lagrangian plane corresponds to a simple 2-vector in the kernel of ω. The kernel of ω is exactly the orthogonal complement of ω^* since

$$\omega(v \wedge w) = 0 \Leftrightarrow (\omega^*, v \wedge w) = 0.$$

Moreover, simple 2*-vectors have zero norm under the inner product* $(,)$*, so Lagrangian planes correspond exactly to isotropic lines.*

Proposition 3. *The stabilizer of a line in V maps to the stabilizer of an isotropic* 2*-plane.*

Proof. *Fix $v \in V$ spanning a line. Consider the set of* 2*-vectors $P = \{v \wedge w | w \in V\}$. This is the set of Lagrangian planes containing v. Stabilizing v implies stabilizing the set of Lagrangian planes containing it, and vice-versa. P is a totally isotropic linear subspace of $\Lambda^2 V$ since $v \wedge w \wedge v \wedge w' = 0$.*

As a corollary, the stabilizer of an isotropic flag maps to the stabilizer of an isotropic flag (it is the intersection of the previous two subgroups).

3 The Einstein Universe and Photon Space

The Einstein Universe is the model for flat conformal Lorentzian geometry. It is the Lorentzian analog of the conformal sphere in Riemannian geometry.

Definition 1. The *Einstein Universe* is the quadric hypersurface $\mathrm{Ein} \subset \mathbb{RP}^4$ given by the zero locus of a signature $(3,2)$ quadratic form.

Proposition 4. *The signature $(3,2)$ bilinear form induces a conformal class of Lorentzian metrics on the Einstein Universe.*

Proof. *Consider the zero locus of the quadratic form $q(x,y,z,u,v) = x^2 + y^2 + z^2 - u^2 - v^2$. Every line through the origin in this locus intersects the radius 2 euclidean ball $x^2 + y^2 + z^2 + u^2 + v^2 = 2$. The points in the intersection satisfy $(x^2 + y^2 + z^2) = (u^2 + v^2) = 1$. The restriction of the bilinear form to this submanifold has signature $(2,1)$. Any other local section of the quotient map $\mathbb{R}^{3,2} \to \mathrm{Ein}$ also induces a bilinear form of the same signature and they all differ by conformal changes.*

The corresponding homogeneous space for $\mathrm{Sp}(4,\mathbb{R})$ is the space of Lagrangians in $\mathbb{R}^4$. Thus the properties of the Einstein universe all have a translation in the language of the Lagrangian Grassmanian.

The basic geometric notion in the Einstein Universe is that of *incidence*. In the $\mathbb{R}^{3,2}$ model, two points (isotropic lines) are incident if they span an isotropic plane. This is equivalent to their inner product vanishing. In the $\mathrm{Sp}(4,\mathbb{R})$ model, this is the notion of two Lagrangian planes intersecting in a line.

A *photon* is a maximal set of pairwise incident points. This means we can represent a photon by an isotropic plane in $\mathbb{R}^{3,2}$ or by a line in $\mathbb{R}^4$.

There is a duality between points and photons, as well as a duality between the projectivized lightcone model and the Lagrangian Grassmannian models. In the projectivized lightcone, two photons intersect in a point if the corresponding isotropic planes intersect in a line. In the dual model, two photons intersect in a point if the corresponding lines in $\mathbb{R}^4$ span a Lagrangian $2-$plane.

The *lightcone* of a point $p \in \mathrm{Ein}$ is the set of all points incident to p. It is topologically a pinched torus. Sometimes we also talk about lightcones as subsets of the space of photons, in which case we mean the set of all photons through p, which is topologically a circle. In the $\mathbb{R}^{3,2}$ model, the lightcone of a point represented by an isotropic line is just the orthogonal subspace to that line, intersected with the null cone in $\mathbb{R}^(3,2)$. In the $\mathbb{R}^4$ model, the lightcone of a Lagrangian P is most conveniently thought of as the set of $1-$dimensional subspaces (photons) in that Lagrangian.

4 Lie Circle Geometry

There is a third and less known model for the Einstein Universe which is convenient for intuition and drawing pictures. It comes from a geometry invented by Sophus Lie called *Lie Sphere Geometry*. In this model, points are represented by oriented circles in the $2-$sphere, which can possibly have zero radius in which case they don't have an orientation.

In this model, the relation of incidence becomes the relation of oriented tangency (contact of order two) between circles. Hence, a photon is a maximal set of pairwise tangent circles, which can be represented as a contact element in S^2. The contact structure on photon space here is just the canonical contact structure on the set of contact elements in any manifold.

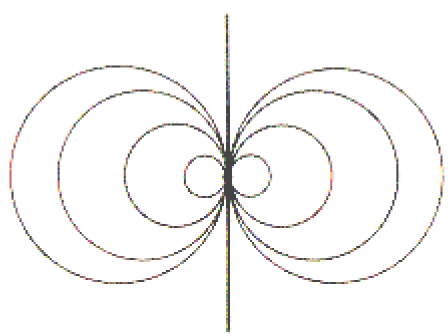

Fig. 1 Points on a common photon in the Lie circle model.

References

References may be *cited* in the text either by number (preferred) or by author/year.[1] The reference list should ideally be *sorted* in alphabetical order – even if reference numbers are used for the their citation in the text. If there are several works by the same author, the following order should be used:

1. all works by the author alone, ordered chronologically by year of publication
2. all works by the author with a coauthor, ordered alphabetically by coauthor
3. all works by the author with several coauthors, ordered chronologically by year of publication.

The *styling* of references[2] depends on the subject of your book:

- The *two* recommended styles for references in books on *mathematical, physical, statistical and computer sciences* are depicted in [1, 2, 3, 4, 5] and [6, 7, 8, 9, 10].
- Examples of the most commonly used reference style in books on *Psychology, Social Sciences* are [11, 12, 13, 14, 15].
- Examples for references in books on *Humanities, Linguistics, Philosophy* are [16, 17, 18, 19, 20].
- Examples of the basic Springer style used in publications on a wide range of subjects such as *Computer Science, Economics, Engineering, Geosciences, Life Sciences, Medicine, Biomedicine* are [21, 22, 24, 23, 25].

1. Broy, M.: Software engineering — from auxiliary to key technologies. In: Broy, M., Dener, E. (eds.) Software Pioneers, pp. 10-13. Springer, Heidelberg (2002)
2. Dod, J.: Effective substances. In: The Dictionary of Substances and Their Effects. Royal Society of Chemistry (1999) Available via DIALOG.
http://www.rsc.org/dose/title of subordinate document. Cited 15 Jan 1999
3. Geddes, K.O., Czapor, S.R., Labahn, G.: Algorithms for Computer Algebra. Kluwer, Boston (1992)
4. Hamburger, C.: Quasimonotonicity, regularity and duality for nonlinear systems of partial differential equations. Ann. Mat. Pura. Appl. **169**, 321–354 (1995)
5. Slifka, M.K., Whitton, J.L.: Clinical implications of dysregulated cytokine production. J. Mol. Med. (2000) doi: 10.1007/s001090000086

6. J. Dod, in *The Dictionary of Substances and Their Effects*, Royal Society of Chemistry. (Available via DIALOG, 1999), http://www.rsc.org/dose/title of subordinate document. Cited 15 Jan 1999
7. H. Ibach, H. Lüth, *Solid-State Physics*, 2nd edn. (Springer, New York, 1996), pp. 45-56
8. S. Preuss, A. Demchuk Jr., M. Stuke, Appl. Phys. A **61**
9. M.K. Slifka, J.L. Whitton, J. Mol. Med., doi: 10.1007/s001090000086
10. S.E. Smith, in *Neuromuscular Junction*, ed. by E. Zaimis. Handbook of Experimental Pharmacology, vol 42 (Springer, Heidelberg, 1976), p. 593

---

[1] Make sure that all references from the list are cited in the text. Those not cited should be moved to a separate *Further Reading* section or chapter.

[2] Always use the standard abbreviation of a journal's name according to the ISSN *List of Title Word Abbreviations*, see http://www.issn.org/en/node/344

11. Calfee, R. C., & Valencia, R. R. (1991). *APA guide to preparing manuscripts for journal publication.* Washington, DC: American Psychological Association.
12. Dod, J. (1999). Effective substances. In: The dictionary of substances and their effects. Royal Society of Chemistry. Available via DIALOG.
http://www.rsc.org/dose/Effective substances. Cited 15 Jan 1999.
13. Harris, M., Karper, E., Stacks, G., Hoffman, D., DeNiro, R., Cruz, P., et al. (2001). Writing labs and the Hollywood connection. *J Film* Writing, 44(3), 213–245.
14. O'Neil, J. M., & Egan, J. (1992). Men's and women's gender role journeys: Metaphor for healing, transition, and transformation. In B. R. Wainrig (Ed.), *Gender issues across the life cycle* (pp. 107–123). New York: Springer.
15. Kreger, M., Brindis, C.D., Manuel, D.M., Sassoubre, L. (2007). Lessons learned in systems change initiatives: benchmarks and indicators. *American Journal of Community Psychology*, doi: 10.1007/s10464-007-9108-14.

16. Alber John, Daniel C. O'Connell, and Sabine Kowal. 2002. Personal perspective in TV interviews. *Pragmatics* 12:257–271
17. Cameron, Deborah. 1997. Theoretical debates in feminist linguistics: Questions of sex and gender. In *Gender and discourse*, ed. Ruth Wodak, 99–119. London: Sage Publications.
18. Cameron, Deborah. 1985. *Feminism and linguistic theory.* New York: St. Martin's Press.
19. Dod, Jake. 1999. Effective substances. In: The dictionary of substances and their effects. Royal Society of Chemistry. Available via DIALOG.
http://www.rsc.org/dose/title of subordinate document. Cited 15 Jan 1999
20. Suleiman, Camelia, Daniel C. OConnell, and Sabine Kowal. 2002. 'If you and I, if we, in this later day, lose that sacred fire...': Perspective in political interviews. *Journal of Psycholinguistic Research*. doi: 10.1023/A:1015592129296.

21. Brown B, Aaron M (2001) The politics of nature. In: Smith J (ed) The rise of modern genomics, 3rd edn. Wiley, New York
22. Dod J (1999) Effective Substances. In: The dictionary of substances and their effects. Royal Society of Chemistry. Available via DIALOG.
http://www.rsc.org/dose/title of subordinate document. Cited 15 Jan 1999
23. Slifka MK, Whitton JL (2000) Clinical implications of dysregulated cytokine production. J Mol Med, doi: 10.1007/s001090000086
24. Smith J, Jones M Jr, Houghton L et al (1999) Future of health insurance. N Engl J Med 965:325–329
25. South J, Blass B (2001) The future of modern genomics. Blackwell, London