SEMISIMPLE LIE GROUPS

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1. Outiline

The goal is to talk about semisimple Lie groups, mainly noncompact real semisimple Lie groups. This is a very broad subject so we will do our best to be concise and cover just what we need to discuss the Cartan decomposition theorem, parabolic subgroups and associated symmetric spaces. To do this we need to first introduce some notation and review a lot of Lie algebra theory, the sources I have used are [1] [3] [5] [4] [6]. Here is a general outline of what will be covered.

1. General set up
2. Real forms, Cartan involutions and Cartan Decomposition Theorem
3. Some facts about the symmetric space G/K.
4. Complex semisimple Lie algebras
5. Restricted roots, Iwasawa Decomposition and real parabolic subgroups

The talk will have many examples even though this write up does not. Very little will be proven.

2. General Set up

Let G be a Lie group, left translation gives rise to a finite dimensional Lie subalgebra of the Lie algebra of vector fields on G with Lie bracket, this the Lie algebra of left invariant vector fields. If $L_g : G \to G$ denotes left translation by $g \in G$ then a vector field $X \in (\Gamma(TG))$ is left invariant if $dL_g(X) = X$ for all $g \in G$, i.e. $\forall g \in G$ and $\forall\ h \in G$, we have $dL_g(X(h)) = X(g \cdot h)$.

The set of left invariant vector fields is closed under Lie bracket, making it into a Lie subalgebra which we denote by $\mathfrak{g} \subset \Gamma(TG)$. Since every left invariant vector field is determined by its value at a single point it follows that $\mathfrak{g} \cong T_gG$ for any $g \in G$, we therefore identify $\mathfrak{g}$ with the tangent space at the identity, $\mathfrak{g} = T_eG$.

The bracket on T_eG is defined by taking the Lie bracket of the associated left invariant vector fields and evaluating it at the identity. With this description of the Lie algebra we get a natural description of the exponential map, $exp : \mathfrak{g} \to G$ by flowing along the left invariant vector fields associated to $\mathfrak{g}$.

2.1. Lie algebras.

This section is mostly from [6]. Given a finite dimensional Lie algebra $\mathfrak{g}$ over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ we get a natural representation of

$$ad : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$$

defined by

$$ad_X(Y) = [X, Y]$$

(the fact that it is a representation follows from the Jacobi identity). The image of ad

$$ad_{\mathfrak{g}} \subset \mathfrak{Der}(\mathfrak{g}) \subset \mathfrak{gl}(\mathfrak{g})$$

is, by definition, the ideal of inner derivations in the Lie algbera $\mathfrak{der}(\mathfrak{g})$ of derivations on $\mathfrak{g}$.

For every $X \in \mathfrak{g}$ we have $exp(ad_X) \subset Aut(\mathfrak{g}) \subset GL(\mathfrak{g})$, the image of exp in $Aut(\mathfrak{g})$ is called the group of inner automorphisms and will be denoted $Int(\mathfrak{g})$. In fact, the Lie subalgebras $ad_{\mathfrak{g}} \subset \mathfrak{der}(\mathfrak{g})$ are the tangent Lie algebras of $Int(\mathfrak{g}) \subset Aut(\mathfrak{g})$.

If G is a Lie group then G acts on itself via inner automorphisms. That is, if $\alpha_g : G \to G$ is defined by

$$\alpha_g(h) = g \cdot h \cdot g^{-1}$$

we get a representation $Ad : G \to Aut(\mathfrak{g}) \subset GL(\mathfrak{g})$ of G in $\mathfrak{g}$ given by $Ad_g = d(\alpha_g)(e)$. The Lie algebra representation corresponding to Ad (i.e. $d(Ad)(e)$) is the previously mentioned ad. The image of Ad, denoted Ad_G, is called the adjoint group of G. It is the Lie group with lie algebra $\mathfrak{g}$ and trivial center.

We will mainly be interested in semisimple Lie algebras and Lie groups. A Lie group is called nilpotent, solvable, reductive, semisimple or simple if its Lie algebra has this property. We begin with a few definitions.

Definition 2.1. A Lie algebra $\mathfrak{g}$ is simple if it is nonabelian and contains no nontrivial ideals.

Semisimple Lie algebras are direct sums of simple Lie algebras. This is usually something that is proven and not taken as a definition. To properly define semisimple Lie algebras we need to first define solvable Lie algebras.

Definition 2.2. Let $\mathfrak{g}$ be a Lie algebra, let $\mathfrak{g}^1 = [\mathfrak{g}, \mathfrak{g}]$ and define $\mathfrak{g}^j = [\mathfrak{g}^{j-1}, \mathfrak{g}^{j-1}]$ inductively, $\mathfrak{g}$ is called solvable if the series

$$\mathfrak{g} \supset \mathfrak{g}^1 \supset \mathfrak{g}^2 \supset \dots$$

terminates.

The standard example of a solvable Lie algebra is the Lie algebra of upper triangular matrices.

Definition 2.3. Let $\mathfrak{g}$ be a Lie algebra, let $\mathfrak{g}_1 = [\mathfrak{g}, \mathfrak{g}]$ and define $\mathfrak{g}_j = [\mathfrak{g}, \mathfrak{g}_{j-1}]$ inductively, $\mathfrak{g}$ is called nilpotent if the series

$$\mathfrak{g} \supset \mathfrak{g}_1 \supset \mathfrak{g}_2 \supset \dots$$

terminates.

The standard example of a nilpotent Lie algebra is the Lie algebra of strictly upper triangular matrices.

Definition 2.4. A Lie algebra $\mathfrak{g}$ is semisimple if it contains no nonzero solvable ideals.

Another useful criterion for semisimplicity was given by Cartan using the Killing form. The representation ad allows us to define an $Int(\mathfrak{g})$ invariant symmetric bilinear form called the Killing form,

$$B_{\mathfrak{g}} : \mathfrak{g} \times \mathfrak{g} \to \mathbb{K},$$

defined by

$$B_{\mathfrak{g}}(X, Y) = Tr(ad_x ad_y).$$

Theorem 2.5. *Lie algebra is semisimple if and only if $B_{\mathfrak{g}}$ is nondegenerate.*

From this we see that semisimple Lie algebras have trivial centers. One key property of semisimple Lie algebra's is that every finite dimensional representation splits as a direct sum of irreducible subrepresentations. The converse is not true, for instance $\mathfrak{g}(n, \mathbb{R})$ has the property that every finite dimensional representation splits as a direct sum of irreducible subrepresentations; but it is not semisimple (it has a center). This leads us to the slightly more general notion of a reductive Lie algebra

Definition 2.6. A Lie algebra is reductive if it is of the form $\mathfrak{z}(\mathfrak{g}) \oplus [\mathfrak{g}, \mathfrak{g}]$. Alternatively, a Lie algebra is reductive if and only if every finite dimensional representation splits as a direct sum of irreducibles.

3. Real forms, Cartan involutions and Cartan Decomposition Theorem

This section is mostly from [5] and [6]. As mentioned at the beginning, we are mainly interested in real semisimple Lie groups and Lie algebra's. However, the structure theory of complex semisimple Lie algebras is simpler and has many nice properties. To go from complex Lie algbera's to real Lie algebras we need to discuss real forms. First, note that if $\mathfrak{g}$ is a real Lie algebra then

$$\mathfrak{g} \otimes \mathbb{C} \cong \mathfrak{g} \oplus i\mathfrak{g}$$

is a complex Lie algebra, note also that $\mathfrak{g}$ is semisimple if and only if $\mathfrak{g} \otimes \mathbb{C}$ is. In the decomposition

$$\mathfrak{g} \otimes \mathbb{C} = \mathfrak{g} \oplus i\mathfrak{g},$$

$\mathfrak{g}$ is the fixed points of the conjugate linear involution of conjugation, this leads us to the definition of a general real form.

Definition 3.1. Let $\mathfrak{g}$ be a complex Lie algebra and

$$\sigma : \mathfrak{g} \to \mathfrak{g}$$

a conjugate linear involution, σ defines a decomposition

$$\mathfrak{g} = \mathfrak{g}^+ \oplus \mathfrak{g}^-$$

of $\mathfrak{g}$ into its $+1$ and -1 eigenspaces. The subalgebra $\mathfrak{g}^+$ is called a real form of $\mathfrak{g}$.

If $\mathfrak{g}^+ \subset \mathfrak{g}$ is the real form coming from involution σ then, since σ is conjugate linear, it has the property $\mathfrak{g}^+ \otimes \mathbb{C} \cong \mathfrak{g}$.

Here are some definitions and facts we will need:

1. A Lie algebra is called compact if and only if it is the Lie algebra of a compact Lie group.
2. It is a theorem that a Lie algebra is compact if and only if it admits an invariant scalar product.
3. In the semisimple case the Killing form, B, is negative definite if and only if the Lie algebra is compact, in which case $-B$ provides an invariant scalar product.
4. Complex semisimple Lie algebra's have two special types of real forms; a compact real form and a split real form both of which are unique up to conjugation.
5. The compact real form is a real form $\mathfrak{g}^+ \subset \mathfrak{g}$ that is a compact Lie algebra.

We will denote the compact real involution by σ, and denote the $+1$ eigenspace of σ by $\mathfrak{u}$. Then

$$\mathfrak{g} = \mathfrak{u} \oplus i\mathfrak{u},$$

and the Killing form is negative definite on $\mathfrak{u}$ and positive definite on $i\mathfrak{u}$. If G is a complex semisimple Lie group with Lie algebra $\mathfrak{g}$, then $exp(\mathfrak{u})$ is a maximal compact subgroup of the complex Lie group G.

3.1. **Cartan Decomposition on Lie algebra level.**

Definition 3.2. Let $\mathfrak{g}$ be a real Lie algebra, an involution $\theta : \mathfrak{g} \to \mathfrak{g}$ is called a Cartan involution if the symmetric bilinear form $B_\theta(X, Y) = -B(X, \theta(Y))$ is positive definite.

Every real semisimple Lie algebra has a Cartan involution, this is a consequence of the existence of compact real forms for complex semisimple Lie algebras. To see this, let $\mathfrak{g}$ be a real semisimple Lie algebra and

$$\mathfrak{g}^{\mathbb{C}} \cong \mathfrak{g} \oplus i\mathfrak{g}$$

be its complexification, let τ be the conjugation on $\mathfrak{g}^{\mathbb{C}}$, giving real form $\mathfrak{g}$. Let $\sigma : \mathfrak{g}^{\mathbb{C}} \to \mathfrak{g}^{\mathbb{C}}$ be a compact real form so that $\tau\sigma = \sigma\tau$, (such a compact form always exists).

Claim 3.3. $(\tau\sigma) = \theta$ *is a Cartan involution when restricted to* $\mathfrak{g}$.

Proof. The involution θ defines an eigenspace decomposition of $\mathfrak{g}^{\mathbb{C}} = \mathfrak{g}^{\mathbb{C}}_+ \oplus \mathfrak{g}^{\mathbb{C}}_-$. Since θ commutes with σ and τ, both real forms $\mathfrak{u}$ and $\mathfrak{g}$ are stable under θ. Thus we have the eigenspace decompositions

$$\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$$

$$\mathfrak{u} = \mathfrak{u}_+ \oplus \mathfrak{u}_-.$$

Call $\mathfrak{g}_+ = \mathfrak{k}$ and $\mathfrak{g}_- = \mathfrak{p}$, note that θ coincides with σ on $\mathfrak{g}$, so $\mathfrak{k} = \mathfrak{u}_+$ and $\mathfrak{p} = i\mathfrak{u}_-$. We have the following decompositions:

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$$

$$\mathfrak{u} = \mathfrak{k} \oplus i\mathfrak{p}.$$

It is now straightforward to check that θ defines a Cartan involution. $\square$

We call a choice of such a involution θ and its corresponding decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ a Cartan Decomposition.

Note for a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ we have

$$[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k} \qquad [\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p} \qquad [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$$

It follows that the Cartan decomposition is orthogonal with respect to both B and B_θ. The Killing form B is negative definite on $\mathfrak{k}$ and positive definite on $\mathfrak{p}$, thus $\mathfrak{k} \subset \mathfrak{g}$ is a maximal compact subalgebra. If G is a semisimple Lie group with Lie algebra $\mathfrak{g}$, then $\exp(\mathfrak{k}) = K \subset G$ is a maximal compact subgroup. Thus a Cartan decomposition gives a Ad_K invariant orthogonal splitting $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$.

3.2. **Cartan Decomposition on Lie group level.** The Cartan decomposition theorem gives a decomposition of a real semisimple Lie Group G with respect to a maximal compact subgroup. We will not prove the theorem, however during the talk we will give some examples.

Theorem 3.4. *Let G be a semisimple Lie group, let θ be a Cartan involution of $\mathfrak{g}$, let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the corresponding Cartan decomposition, and let K be the analytic subgroup of G with Lie algebra $\mathfrak{k}$. Then*

1. *There exists a Lie group involutative automorphism Θ of G with differential θ.*
2. *The subgroup of G fixed by Θ is K*
3. *The mapping $K \times \mathfrak{p} \to G$ defined by $(k, X) \mapsto (kexp(X))$ is a diffeomorphism.*
4. *K is closed and contains the center of G.*

Note that this theorem generalizes the polar decomposition of invertible matrices. Notice also that this decomposition is not a decomposition into subgroups, as $exp(\mathfrak{p}) \subset G$ is not a subgroup.

4. The symmetric space G/K

This brief section is mostly from [1]. Let $H \subset G$ be a closed subgroup and suppose $\mathfrak{g}$ admits a decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$$

that is Ad_H invariant and orthogonal with respect to a G-invariant metric. By the results in the previous section, if $K \subset G$ is a maximal compact subgroup then $\mathfrak{g}$ admits such a decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$.

The Lie group G is the total space of a principal H bundle $G \to G/H$. If

$$\omega_{MC}^{G} \in \Omega^1(G, \mathfrak{g})$$

is the (left) Maurer-Cartan form of G and

$$P_{\mathfrak{h}} : \mathfrak{g} \to \mathfrak{h}$$

is the projection onto $\mathfrak{h}$ then

$$P_{\mathfrak{h}} \circ \omega_{MC}^{G} = \theta^c \in \Omega^1(G, \mathfrak{h})$$

defines a canonical connection one form on the principal H bundle $G \to G/H$.

The tangent bundle TG/H is canonically isomorphic to $G \underset{Ad_H}{\times} \mathfrak{m}$. Since TG/H is an associated bundle of $G \to G/H$, the canonical connection one form θ^c induces a covariant derivative ∇^c on TG/H. Furthermore, by construction, all G invariant tensors on G/H are covariantly constant with respect to ∇^c. A G-invariant metric is such a tensor, hence ∇^c is automatically a metric connection for any G-invariant metric on G/H.

The question of whether ∇^c is the Levi-Civita connection of a G-invariant metric requires examining the torsion of ∇^c. It turns out that the torsion tensor $T(\nabla^c)$ vanishes if and only in

$$[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$$

that is, if and only if G/H is a symmetric space, see [1] for more details. As a result, a lot of the Riemannian geometry of symmetric spaces can be understood from the Lie theory.

By the Cartan decomposition theorem, G/K is a symmetric space. The Levi-Civita connection of any G invariant metric is therefore the one induced by the canonical connection on $G \to G/K$. One place this is useful (for higher Teichmüller Theory) is in understanding the harmonic metric equations associated to a reductive representation. If Σ is a closed surface, the harmonic map associated to a representation

$$\rho : \pi_1(\Sigma) \to G$$

in Corlette's theorem [2] is a map $h : \widetilde{\Sigma} \to G/K$.

5. Complex semisimple Lie algebras

This section is mostly from [3]. For this section $\mathfrak{g}$ will be a complex semisimple Lie algebra. One key property of such a $\mathfrak{g}$ is: there exists a maximally commutative subalgebra $\mathfrak{h} \subset \mathfrak{g}$ whose adjoint action on $\mathfrak{g}$ is diagonal and is unique up to conjugation. By diagonal, we mean the Lie algebra representation $ad : \mathfrak{h} \rightarrow \mathfrak{gl}(\mathfrak{g})$ decomposes $\mathfrak{g}$ into a direct sum irreducible representations called weight spaces. A non zero weight for the adjoint action is called a root; the roots form a finite subset $\Delta \subset \mathfrak{h}^*$ and $\mathfrak{h}$ acts on itself with weight 0. We have

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha.$$

where $\mathfrak{g}_0 = \mathfrak{h}$. The spaces $\mathfrak{g}_\alpha$ are the root spaces, they are defined by

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} | ad_H(X) = \alpha(H)X \quad \forall H \in \mathfrak{h}\}.$$

Some important properties of the root space decomposition include:

1. $\mathfrak{h}(\mathbb{R}) = \{h \in \mathfrak{h} | \alpha(h) \in \mathbb{R} \quad \forall \alpha \in \Delta\}$ is a real form of $\mathfrak{h}$.
2. If $\alpha \in \Delta$ then $n\alpha \in \Delta$ if and only if $n = -1$.
3. If $\alpha \in \Delta$ then $dim(\mathfrak{g}_\alpha) = 1$.
4. If $\alpha, \beta \in \Delta$ then $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$, in particular $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \subset \mathfrak{h}$ and $< \mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}, [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] >$ form a subalgebra isomorphic to $\mathfrak{sl}(2, \mathbb{C})$.
5. $< \Delta >= \mathfrak{h}^*$
6. The Killing form satisfies $B(u, v) = 0$ for two root vectors u, v corresponding to roots $\alpha, \beta \in \Delta$ with $\alpha + \beta \neq 0$. We also have that $B|\mathfrak{h}$ and $B|(\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha})$ are nondegenerate and $B|\mathfrak{h}(\mathbb{R})$ is real and positive definite.

The dimension of a Cartan subalgebra is called the rank of $\mathfrak{g}$. Property 2 tells us that if we choose a linear function $l : \mathfrak{h}^* \rightarrow \mathbb{R}$ with the property $l(\Delta) \neq 0$, we have a splitting of Δ into positive and negative roots, $\Delta = \Delta_+ \cup \Delta_-$, here

$$\Delta_+ = \{\alpha \in \Delta | l(\alpha > 0\}.$$

A positive root $\alpha \in \Delta_+$ is called simple if it can not be written as a linear combination (with positive coefficients) of elements of Δ_+, we denote the set of simple roots by $\Pi = \{\alpha_1, \ldots, \alpha_n\} \subset \Delta_+$. It is clear that all elements of Δ are linear combinations of elements of Π, thus, by property 5 above we know that $< \Pi >= \mathfrak{h}^*$. In fact Π forms a basis for $\mathfrak{h}^*$, and provide us with a canonical basis of

$$\mathfrak{g} =< h_{\alpha_i}, e_\alpha, \tilde{e}_\alpha > .$$

Here $h_{\alpha_i} \in \mathfrak{h}$ is obtained from the isomorphism $\mathfrak{h}^* \rightarrow \mathfrak{h}$ given by the Killing form and $e_\alpha \in \mathfrak{g}_\alpha$ and $\tilde{e}_\alpha \in \mathfrak{g}_{-\alpha}$ for $\alpha \in \Delta_+$, are defined so that

$$< e_\alpha, \tilde{e}_\alpha, [e_\alpha, \tilde{e}_\alpha] = h_\alpha > \cong \mathfrak{sl}(2, \mathbb{C}).$$

This leads us to the definitions of standard complex Borel and parabolic subalgebras.

Definition 5.1. The standard Borel subalgebra (or minimal parabolic) with respect to a choice Cartan subalgebra and positive roots, is defined by

$$\mathfrak{b} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta_+} \mathfrak{g}_\alpha.$$

A standard parabolic subalgebra is any subalgebra containing $\mathfrak{b}$, a parabolic subalgebra is determined by a choice of subset of $-\Pi$.

General Parabolic subalgebras are conjugates of standard ones and parabolic subgroups are just exponentials of parabolic subalgebras. We will discuss these more in the real case below.

6. Restricted roots, Iwasawa Decomposition and real parabolic subgroups

This section is mostly from [5]. The goal of this section is to define parabolic subgroups of real semisimple Lie groups. To do this we need to introduce restricted roots and the Iwasawa Decomposition. Parabolic subgroups are defined by the property that G/P is compact. The main complication is that for a real semisimple Lie algebra the maximal abelian subalgebra that acts diagonally on $\mathfrak{g}$ may be too small to complexify to be a Cartan subalgebra of $\mathfrak{g}^{\mathbb{C}}$. For split real Lie groups, this complication disappears.

We start with a real semisimple Lie algebra $\mathfrak{g}$ with Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Let $\mathfrak{a}$ be the maximal abelian subspace of $\mathfrak{p}$, (note $exp(\mathfrak{a}) \cong (R^+)^{dim \mathfrak{a}}$) since the adjoint of $X \in \mathfrak{g}$ with respect to the postive definite form B_θ satisfies $(ad_X)^* = -ad_{\theta X}$, we see that X is self adjoint if and only if $X \in \mathfrak{p}$. Thus the set $\{ad_H | H \in \mathfrak{a}\}$ is a commiting family of self adjoint transformations of $\mathfrak{g}$. From this we conclude $\mathfrak{g}$ is a direct sum of simultaneous eigenspaces, all the eigenvalues being real.

For $\lambda \in \mathfrak{a}^*$ we have

$$\mathfrak{g}_\lambda = \{X \in \mathfrak{g} | ad_H(X) = \lambda(H)X \quad \forall H \in \mathfrak{a}\},$$

if $\lambda \neq 0$ and $\mathfrak{g}_\lambda \neq 0$ then λ is called a restricted root. Let Σ denote the set of restricted roots.

The Lie algebra $\mathfrak{g}$ admits an orthogonal decomposition $\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\lambda \in \Sigma} \mathfrak{g}_\lambda$ satisfying the following:

1. $[\mathfrak{g}_\lambda, \mathfrak{g}_\mu] \subset \mathfrak{g}_{\lambda+\mu}$
2. $\theta \mathfrak{g}_\lambda = \mathfrak{g}_{-\lambda}$ hence $\lambda \in \Sigma \implies -\lambda \in \Sigma$.
3. $\mathfrak{g}_0 = \mathfrak{a} \oplus \mathfrak{m}$ orthogonally, where $\mathfrak{m} = Z_{\mathfrak{k}}(\mathfrak{a})$.

The dimension of $\mathfrak{a}$ is called the real rank of $\mathfrak{g}$, if the real rank is the same as the rank of $\mathfrak{g}$ then $\mathfrak{g}$ is a split real form. For split real forms the dimensions of $\mathfrak{g}_\lambda$ are all 1.

6.1. **Iwasawa Decomposition.** Choose a notion of positivity for the restricted roots Σ and let Σ^+ be the positive restricted roots. Define $\mathfrak{n} = \bigoplus_{\lambda \in \Sigma^+} \mathfrak{g}_\lambda$, this is a nilpotent Lie algebra, the Iwasawa decomposition theorem on the level of Lie algebra's is the following.

Theorem 6.1. *In the notation above, $\mathfrak{g}$ is a vector space direct sum $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$. Here $\mathfrak{a}$ is abelian, $\mathfrak{n}$ is nilpotent, $\mathfrak{a} \oplus \mathfrak{n}$ is solvable subalgebra of $\mathfrak{g}$ and $[\mathfrak{a} \oplus \mathfrak{n}, \mathfrak{a} \oplus \mathfrak{n}] = \mathfrak{n}$.*

There is an analogous theorem on the level of Lie Groups.

6.2. **Parabolic subalgebras and subgroups.** A parabolic subgroup of a real semisimple Lie group is the exponential of a parabolic subalgebra, $Q \subset G$ is a parabolic subgroup if and only if G/Q is compact and $\mathfrak{q} \subset \mathfrak{g}$ is a parabolic subalgebra if and only if $\mathfrak{q}^{\mathbb{C}} \subset \mathfrak{g}^{\mathbb{C}}$ is a parabolic subalgebra. Just as in the set up for complex semisimple Lie algebras, we can choose a set of positive restricted roots $\Sigma^+ \subset \Sigma$ and consider the corresponding simple restricted roots.

Definition 6.2. The mininmal parabolic subalgebra (with respect to the choices made) is the subalgebra

$$\mathfrak{b} = \mathfrak{g}_0 \oplus \bigoplus_{\lambda \in \Sigma^+} \mathfrak{g}_\lambda.$$

A general minimal parabolic is one which is conjugate to $\mathfrak{b}$.

Definition 6.3. A parabolic subalgebra is a subalgebra containing a minimal parabolic.

Let $\Pi \subset \Sigma^+$ be the subset of simple restricted roots, up to conjugation the parabolic subalgebras of $\mathfrak{g}$ are characterized by subsets $\Pi' \subset \Pi$. To see this consider

$$\Gamma = \Sigma^+ \cup \{\beta \in \Sigma | \beta \in span(\Pi')\},$$

we get the following parabolic subalgebra from Γ

$$\mathfrak{q} = \mathfrak{g}_0 \oplus \bigoplus_{\lambda \in \Gamma} \mathfrak{g}_\lambda.$$

By flipping our notion of positive to roots, (i.e. the negative roots before are now the positive roots) we have a similar minimal parabolic subalgebra

$$\mathfrak{b}^- = \mathfrak{g}_0 \oplus \bigoplus_{\lambda \in \Sigma^-} \mathfrak{g}_\lambda.$$

Similarly corresponding to $\Pi' \subset \Pi$ we have $-\Pi' \subset -\Pi$ and

$$-\Gamma = \Sigma^- \cup \{\beta \in \Sigma | \beta \in span(-\Pi')\},$$

and the corresponding parabolic subalgebra

$$\mathfrak{q}^- = \mathfrak{g}_0 \oplus \bigoplus_{\lambda \in -\Gamma} \mathfrak{g}_\lambda.$$

For notational clarity we write $\mathfrak{q} = \mathfrak{q}^+$ and $\mathfrak{b} = \mathfrak{b}^+$, the intersection

$$\mathfrak{q}^+ \cap \mathfrak{q}^- = \mathfrak{g}_0 \bigoplus_{\{\lambda \in \Sigma | \lambda \in span(\Pi')\}} \mathfrak{g}_\lambda$$

is called the Levi subalgebra of $\mathfrak{q}$. By exponentiating we get two parabolics $Q^+ = exp(\mathfrak{q}^+)$ and $Q^- = exp(\mathfrak{q}^-)$ called opposite parabolics, the intersection $Q^+ \cap Q^-$ is called a Levi subgroup associated to Q.

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