9.1/9.2 Main ideas

1. What is a Fourier series? And why might you want one?

2. What does it mean for a function to be periodic?

3. How do you find the Fourier series of a function \( f \) of period \( P = 2L \)?

4. What do Fourier series converge to?

5. What does it mean for a function to be even? Odd? What does this mean about the integral \( \int_{-a}^{a} f(t) \, dt \)?

6. Some new favorite trig identities:
   
   (a) \( \sin(A \pm B) = \)
   
   (b) \( \cos(A \pm B) = \)
   
   (c) \( \cos A \cos B = \)
   
   (d) \( \sin A \cos B = \)
   
   (e) \( \sin A \sin B = \)
   
   (f) \( \sin^2 A = \)
   
   (g) \( \cos^2 A = \)
Example

Find the Fourier series and the function the Fourier series converges to of the periodic function with one full period given by

\[ f(x) = \begin{cases} 
-1 & -2 < x \leq 0 \\
1 & 0 < x \leq 2 
\end{cases} \]
Practice Problems

1. Find the values of $a_0, a_n, b_n$ in the Fourier series for the following functions without calculating any integrals. You will need some trig identities.

   (a) $f(x) = \sin x$
   (b) $f(x) = 4 \sin^2(3x) + 4 \cos^2(5x)$
   (c) $f(x) = 6 \sin(5x) \cos(2x)$
   (d) $f(x) = 3 \cos(x) \cos(7x)$

2. Find the Fourier series of $f(t) = \begin{cases} 0 & -\pi \leq t < 0 \\ \sin t & 0 \leq t < \pi \end{cases}$.

3. Let $f(t) = \begin{cases} -t & -3 < t < 0 \\ t^2 & 0 \leq t < 3 \end{cases}$ where $f(t + 6) = f(t)$ for all $-\infty < t < \infty$.

   (a) Graph two periods of $f(t)$
   (b) Find the Fourier Series of $f$.
   (c) Sketch the graph of the function to which your series converges to for $-9 \leq t \leq 9$

4. The Fourier series for $f(t) = \frac{3t^2 - 6\pi t + 2\pi^2}{12}$ for $0 < t < 2\pi$ is $\sum_{n=1}^{\infty} \frac{\cos nt}{n^2}$.

   (a) Derive this result
   (b) Graph the period $2\pi$ function to which the series converges
   (c) Use this to calculate $\sum_{n=1}^{\infty} \frac{\cos n}{n^2}$
Some Random Exam Practice

1. Find the general solution to \( y'' - 4y' - 12y = 3e^{5t} + \sin(2t) + te^{4t} \)

2. Find the general solution to \( 4y'' + 16y' + 17y = e^{-2t} \sin\left(\frac{t}{2}\right) + 6t \cos\left(\frac{t}{3}\right) \)

3. Find a general solution to \( 2y'' + 18y = 6 \tan(3t) \). **What method is needed?**

4. (On this problem, you can use a calculator to do some arithmetic. You will also need to watch your units). A 3 kg object (large chickens are apparently about 3 kg) is attached to a spring and will stretch the spring 392 mm by itself (with gravity). A dashpot will exert a force of 45 Newtons when the velocity is 50 cm/sec and an external force

   (a) Set-up the differential equation for this oscillator.

   (b) The object (chicken) is initially displaced 20 cm downward from its equilibrium position and given a velocity of 10 cm/sec upward. What are the initial conditions? (You should probably not try this at home)

   (c) Find the solution to this oscillator.

   (d) What is the long-term behavior (as \( t \to \infty \)) of this solution? Do the initial conditions matter?

5. Suppose \( f \) is periodic and a full period is

\[
 f(t) = \begin{cases} 
 0 & -2\pi < t < 0 \\
 \sin(t) & 0 < t < 2\pi 
\end{cases}.
\]

Determine the Fourier series and the function to which the Fourier series converges.

6. If you have time, derive the new favorite trig identities.

7. Your exam will almost certainly be 1 Fourier series, 3 undetermined coefficients, variation of parameters, and/or oscillator, and 1 eigenvalue problem. Start thinking of questions on these **now**. Start seeking help for any of these topics if you need it **now**.

**An Ending Thought:** *Half of the failures in life come from pulling one’s horse when he is leaping.*

– Thomas Hood