Main Ideas
Section 9.3

• When do you find a cosine Fourier series of \( f(t) \)?

• When do you find a sine Fourier series of \( f(t) \)?

• When do you use a cosine Fourier series for a BVP?

• When do you use a sine Fourier series for a BVP?

• Comment: you multiply your Fourier series integral by 2 when you are integrating over half the period rather than the full period because the function is even.

Section 9.4

• When does resonance occur for a forced spring?

Section 9.5

• What differential equation models a heated rod?

• What do you guess as your solution?

• What is the solution to \( X'' + \lambda X = 0 \) and \( X(0) = X(L) = 0 \)?

• What is the solution to \( X'' + \lambda X = 0 \) and \( X'(0) = X'(L) = 0 \)?

• What is the solution to \( T' + \lambda T = 0 \)?

• How do you deal with the condition with \( u(x, 0) \)? The condition with \( u_t(x, 0) \)? Note: there are two possible options.
Section 9.6

- What is the solution to $T'' + \lambda T = 0$?

Section 9.7

- Comment: if you have two conditions with $X$, put all the constants with $Y$ and then set both equal to $-\lambda$. If you have two conditions with $Y$, put all the constants with $Y$ and then set both equal to $-\lambda$.

- When do you use $e^{\alpha x} + e^{-\alpha x}$?

- When do you use $\sinh(\alpha x) + \cosh(\alpha x)$?

Practice Problems

1. Find the Fourier series for $f(t)$ assuming $f$ is an even function of period $2L$ with the half period given by $f(t) = Ct(L - t)$. Note $C$ is a constant.

   \[
   \int_0^L Ct(L - t) \cos \left( \frac{n\pi t}{L} \right) dt = \begin{cases} \frac{2CL^3}{n^2\pi^2} & n \text{ even} \\ 0 & n \text{ odd} \end{cases}
   \]

2. Find the Fourier series for $f(t)$ assuming $f$ is an odd function of period $2L$ with the half period given by $f(t) = Ct(L - t)$.

   \[
   \int_0^L Ct(L - t) \sin \left( \frac{n\pi t}{L} \right) dt = \begin{cases} \frac{4CL^3}{n^3\pi^3} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}
   \]

3. Find a formal Fourier series solution to $x'' + 2x = 4t(3 - t)$, $x(0) = x(3) = 0$.

4. Find all positive values of $\alpha$ such that the pure resonance will occur in the system $x'' + \alpha^2 x = F(t)$ where $F(t)$ is the even function of period $2\pi$ with $F(t) = -2t(\pi - t)$ for $0 < x < \pi$.

5. Find the steady periodic solution $x_{sp}(t)$ to $x'' - 4x = F(t)$ where $F(t)$ is the even function of period $2\pi$ with $F(t) = 5t(\pi - t)$.

6. Practice by inspection. A function and a condition are given. Find the coefficients in the function such that the condition holds. Then find the function without a sum ($n$ should be nowhere in your answer).

   (a) $u(x, t) = \sum_n A_n e^{-4nt} \cos(nx)$, $u(x, 0) = 3\cos x - 2\cos(5x)$

   (b) $u(x, t) = \sum_n A_n e^{-4nt} \cos \left( \frac{n\pi x}{L} \right)$, $u(x, 0) = 3\cos x - 2\cos(5x)$

   (c) $u(x, t) = \sum_n A_n e^{-4(nt+1)} \cos(nx)$, $u(x, 0) = 3\cos x - 2\cos(5x)$

   (d) $u(x, t) = \sum_n A_n e^{-4nt} \cos(nx)$, $u_t(x, 0) = 3\cos x - 2\cos(5x)$
7. (a) Solve the BVP \( u_t = 3u_{xx}, \ 0 < x < 7, \ t > 0, \ u(0, t) = u(7, t) = 0 \)
   i. \( u(x, 0) = x(x - 7) \)
   ii. \( u_t(x, 0) = x(x - 7) \).
(b) Solve the BVP \( u_t = 3u_{xx}, \ 0 < x < 7, \ t > 0, \ u_x(0, t) = u_x(7, t) = 0 \)
   i. \( u(x, 0) = x(x - 7) \)
   ii. \( u_t(x, 0) = x(x - 7) \). (is this possible? Why or why not?)
(c) How does \( u(0, t) \) vs \( u_x(0, t) \) impact the answer? How about \( u(x, 0) \) vs \( u_t(x, 0) \)?
8. (a) Solve the BVP \( y_{tt} = 4y_{xx}, \ 0 < x < 2, \ t > 0, \ y(0, t) = y(2, t) = 0 \)
   i. \( y(x, 0) = 0, \ y_t(x, 0) = 2x(2 - x) \)
   ii. \( y(x, 0) = 3x(2 - x), \ y_t(x, 0) = 0 \)
   iii. \( y(x, 0) = 3x(2 - x), \ y_t(x, 0) = 2x(2 - x) \)
(b) Why does it make sense that the answer to (c) is the sum of the answers to (a) and (b)?
9. For the following problems, find a solution to the Dirichlet problem for the rectangle \( 0 < x < 3, \ 0 < y < 4 \) consisting of Laplace’s Equation \( u_{xx} + u_{yy} = 0 \) and
   (a) \( u(x, 0) = u(x, 4) = u(0, y) = 0, \ u(3, y) = 2y(4 - y) \)
   (b) \( u(x, 0) = u(x, 4) = u(3, y) = 0, \ u(0, y) = 2y(4 - y) \)
   (c) \( u_y(x, 0) = u_y(x, 4) = u(0, y) = 0, \ u(3, y) = 2y(4 - y) \)
   (d) How are these answers different? What causes the difference(s)?
10. Find a solution to the Dirichlet problem for the semi-infinite strip \( x > 0, \ 0 < y < 4 \) consisting of Laplace’s Equation \( u_{xx} + u_{yy} = 0, \ u(x, y) \) is bounded as \( x \to \infty \) and
    (a) \( u(x, 0) = u(x, 4) = 0, \ u(0, y) = 2y(4 - y) \)
    (b) \( u_y(x, 0) = u_y(x, 4) = 0, \ u(0, y) = 2y(4 - y) \)
    (c) How are these answers different from each other and from the problems in (9)?
        What causes that difference?

An Ending Thought: They can because they think they can.

– Virgil
1. Paying attention to resonance, find a formal Fourier series and the general solution to

\[ x'' + 9x = \sum_{n \text{ even}} \frac{1}{n^2} \cos \left( \frac{nt}{2} \right). \]

2. Do you need sine or cosine Fourier series for the RHS in order to solve the following differential equations?

(a) \( x'' + 3x = F(t) \) where \( F(t) \) is the even function of period 8 and \( F(t) = t \) for \( 0 \leq t \leq 4 \)
(b) \( x'' + 3x = t^2; x(0) = x(4) = 0 \)
(c) \( x'' + 3x = t; x'(0) = x'(4) = 0 \)
(d) \( x'' + 3x = F(t) \) where \( F(t) \) is the odd function of period 8 and \( F(t) = t \) for \( 0 \leq t \leq 4 \)
(e) \( x'' + 3x = t; x(0) = x(4) = 0 \)
(f) \( x'' + 3x = t^2; x'(0) = x'(4) = 0 \)

3. For what values of \( k \) does resonance occur in

\[ x'' + kx = \sum_{n \text{ even}} 500 \sin \left( \frac{nt}{5} \right)? \]

4. Solve the following for \( y(x,t) \):

\[ y_{tt} = 4y_{xx} \quad y(0,t) = y(1,t) = 0 \]
\[ y(x,0) = 0 \quad y_t(x,0) = \sum_{n \text{ odd}} \frac{3}{n^2 \pi^2} \sin(n\pi x) \]

5. Practice whatever you need to practice. There are plenty of review problems on past worksheets.

An Ending Thought: If you really want to do something, you’ll find a way. If you don’t, you’ll find an excuse

– Jim Rohn