1. Writing as a single oscillation: \( y_p = -5 \cos 4t + 12 \sin 4t \)

2. You practice. Write in the form \( y = C \cos(\omega t - \alpha) \)
   
   (a) \( y = 3 \cos 2t + 4 \sin 2t \)  
   (b) \( y = 15 \cos 6t - 8 \sin 6t \)  
   (c) \( y = -3 \sin 2t \)  
   (d) \( y = -2 \cos t - 3 \sin t \)

3. Spring vocabulary
   
   (a) Springs are modeled by \( mx'' + cx' + kx = F(t) \). What do \( m, c, k \) and \( F \) represent?  
   (b) When (in the differential equation) does a spring have free motion? What about forced motion?  
   (c) What kind of system do we call \( c = 0 \)? What about \( c \neq 0 \)?

4. Steps for solving spring problems (these are in order). This is FYI, not a question
   
   (a) Solve the free motion equation using “the \( r \) thing”. This is called either \( x_c \) or \( x_{tr} \)  
   (b) Find a solution to the forced equation using “the method of undetermined coefficients”. This is called either \( x_p \) or \( x_{sp} \)  
   (c) Use any initial conditions  
   (d) Write as a single oscillation  
   (e) Describe the long-term behavior  
   (f) Answer all parts of the question asked
5. Types of damping:

(a) i. Solve $x'' + 2x' + 2x = 0$. This is underdamped. What does the graph look like?
   ii. Solve $x'' + 2x' + x = 0$. This is critically damped. What does the graph look like?
   iii. Solve $x'' + 3x' + 2x = 0$. This is overdamped. What does the graph look like?

(b) Regardless of the type of damping, what is the long-term behavior of the solution? Why does this make sense physically? Do the initial conditions matter for the long-term behavior?

(c) Looking at the three examples above, what determines whether you are under-damped, critically damped, and overdamped?

6. Forced, undamped motion:

(a) Solve

   i. $x'' + x = \cos 2t$. ii. $x'' + x = \cos t$.

(b) Which of the above examples is in resonance? Sketch the graph in that case.

(c) When does resonance occur in general?

(d) For what values of $k$ will the following systems be in resonance?

   i. $x'' + k^2x = \cos 4t$  ii. $x'' + 4x = \cos kt$

7. Forced damped motion: $x'' + cx' + kx = F(t)$

(a) We call the solution to the associated free equation $x_{tr}$ for transient solution. What is the long-term behavior of this solution?

(b) We call the particular solution $x_{sp}$, for steady periodic solution. What is the long-term behavior of this solution?

(c) Do the initial conditions change $x_{tr}$ or $x_{sp}$? Therefore, do initial conditions change the long-term behavior of the solution? (Note: even though the initial conditions only change one, you must calculate the constants using $x_{tr} + x_{sp}$!)

8. There are practice problems (with admittedly annoying algebra) on last week’s worksheet. Find them if you want more practice.

**An Ending Thought:** Success should not go to head and failure should not go to heart.

− Tamil proverb
1. Let’s first explore boundary value problems. These have “boundary conditions” at two different points, as opposed to “initial conditions.”

(a) Find solutions, if they exist, to the following problems

i. \( x'' + x = 0; x(0) = 0; x(1) = 0 \)
ii. \( x'' + x = 0; x(0) = 0; x(\pi) = 0 \)
iii. \( x'' + x = 0; x(0) = 0; x(\pi) = 1 \)

(b) Looking at the examples above, do solutions always exist to BVPs? If they do exist, are they unique?

(c) For what values of \( k \) do solutions exist for \( x'' + x = 0; x(0) = 0, x(k) = 0 \)?

2. Recall some facts we will need a lot in 3.8. Solve for \( k \) (list all values)

(a) \( \sin k = 0 \)  
(b) \( \cos k = 0 \)  
(c) \( e^k - 1 = 0 \)

3. Eigenvalue problems are solving infinitely many BVP at the same time!

(a) Solve

i. \( x'' = 0 \)  
ii. \( x'' + x = 0 \)  
iii. \( x'' - x = 0 \)

(b) Now suppose \( \lambda \) is a real number. We are going to generalize the above examples. Suppose \( x'' + \lambda x = 0 \). Find the solutions (\( \lambda \) will appear) in the following cases.

i. \( \lambda = 0 \)  
ii. \( \lambda > 0 \)  
iii. \( \lambda < 0 \)

(c) Now suppose \( x(0) = 0, x(3) = 0 \). What values of \( \lambda \) can give us solutions? (We call these eigenvalues) What are those solutions? (We call these eigenfunctions). You need to check all three cases above.

(d) What are the eigenvalues and eigenfunctions if \( x'(0) = 0, x'(4) = 0 \)?

(e) If \( x(0) = 0, x'(2) = 0 \)?

4. A problem from your (ungraded) homework: Find the eigenvalues and eigenfunctions of

\( y'' + \lambda y = 0; y'(0) = 0; y(1) + y'(1) = 0 \)

5. And another: \( y'' + 2y' + \lambda y = 0, y(0) = y(1) = 0 \)

**An Ending Thought:** There is a light at the end of every tunnel, just pray it’s not a train.

– Author Unknown