1. Equilibrium solutions

(a) What does it mean for a solution to be an equilibrium solution?

(b) This will require you to think. Suppose $\frac{dx}{dt} = f(x,t)$. When do you get equilibrium solutions? Why?

(c) Generally, can you cross equilibrium solutions? Why or why not?

2. Stability

(a) What are the three types of equilibrium solutions? Draw a picture of each. There are actually four pictures.

(b) Determine the critical points and their stability for each of the following differential equations. Draw a phase diagram for each.

i. $x' = 7x - x^2 - 10$

ii. $x' = (x - 2)^2$

iii. $x' = x^2(x^2 - 4)$

3. Consider the two differential equations

$$\frac{dx}{dt} = (x - a)(x - b)(x - c)$$

and

$$\frac{dx}{dt} = (a - x)(x - b)(x - c)$$

with $a < b < c$. Determine the stability of each critical point. Without attempting to solve the differential equations, make rough sketches of typical solution curves.

4. The next questions are about constant coefficient homogeneous linear differential equations.

(a) What do they look like?

(b) Put the steps for solving them in order

___ Find the characteristic polynomial
___ Sum up the solutions
___ Find the roots
___ Determine the appropriate solution for each root

(c) What does it mean for a differential equation to be $n$th order? How many solutions do you need? How many constants do you have?
5. Distinct real roots

(a) If you get \( n \) distinct real roots \( r_1, r_2, ..., r_n \) what is the general solution?

(b) Practice

i. \( y'' + y' - 12y = 0 \)  
ii. \( y''' - y'' - 14y' + 24y = 0 \)

6. If you have a repeated root, you don’t get \( n \) different functions. We need to fix that.

(a) Consider \( y'' - 2y' + y = 0 \). What is the multiplicity of the repeated root? Show that \( e^x \) and \( xe^x \) are solutions.

(b) Consider \( y''' - 3y'' + 3y' - y = 0 \). What is the multiplicity of the repeated root? Show that \( e^x, xe^x, \) and \( x^2e^x \) are solutions.

(c) If you have a repeated root, \( r \), of multiplicity \( k \), what \( k \) functions do you think are solutions?

(d) Practice

i. \( 2y'' - 12y' + 18y = 0 \)  
ii. \( y''' - 2y'' - 15y' + 36y = 0 \) (\( r = -4 \) is a root and then long divide)

7. Complex roots

(a) When you have roots \( a \pm bi \) what solutions do you get?

(b) What do you think you do if you have \( a \pm bi \) repeated?

(c) Practice

i. \( y'' + y = 0 \)  
ii. \( y'' - 2y' + 5y = 0 \)  
iii. \( y''' - 5y'' + 8y' - 6y = 0 \) (try \( r = 3 \))  
iv. \( y^{(4)} + 2y'' + y = 0 \)

8. You now know how the roots contribute to the general equation. You can use this information to go backwards. What is the differential equation that \( y = c_1e^{3x} + c_2xe^{3x} + c_3 \cos 2x + c_4 \sin(2x) \) is the general solution to? Hint: start with determining the characteristic equation.
9. A large salt tablet is dropped into a beaker of methanol. As the salt dissolves in the methanol, the number \( x(t) \) of grams of the salt in a solution after \( t \) seconds satisfies the differential equation \( \frac{dx}{dt} = 0.8x - 0.004x^2 \).

(a) What is the maximum amount, \( M \) of the salt that will ever dissolve in the methanol?

(b) Let’s think about what it means physically.
   i. What happens when \( x > M \)? Does this make sense physically?
   ii. What happens when \( 0 < x < M \)? Does this make sense physically?
   iii. What happens when \( x < 0 \)? Does this make sense physically?

(c) If \( x = 50 \) when \( t = 0 \), how long will it take for an additional 50 g of salt to dissolve?

10. Misc exam 1 practice. Solve the following equations. Ones with (*) have tricky integrals in their solutions but you should be able to start them. Implicit solutions are ok on ones with (♣).

(a) \( xy' = y(\ln x - \ln y) \) \hspace{1cm} (g) \( 3y + x^4y' = 2xy \)
(b) \( (*) (e^x + e^{-x})y' = y^2 \) \hspace{1cm} (h) \( y' = x^2 - 2xy + y^2 \)
(c) \( (*) y' = \sqrt{x + 2y} \) \hspace{1cm} (i) \( y' = 3(y + 7)x^2 \)
(d) \( (*) y^3y' = (y^4 + 1) \cos x \) \hspace{1cm} (j) \( 2y''' - y'' - 5y' - 2y = 0 \) (hint: -1)
(e) \( (*) y' = \frac{y - x}{y + x} \) \hspace{1cm} (k) \( 2y + (x + 1)y' = 3x + 3 \)
(f) \( (*) yy'' + (y')^2 = yy' \) \hspace{1cm} (l) \( y^{(4)} = 16y \)

An Ending Thought: Nothing great was ever achieved without enthusiasm.

– Ralph Waldo Emerson