1. State the hypotheses and conclusion of the Existence and Uniqueness Theorem.

2. Consider the IVP \( \frac{dy}{dx} = y^{2/3} \) with the initial value \( y(a) = b \).
   (a) Use the Existence and Uniqueness Theorem and determine for what points might be the solution curves for fail to exist or be unique.
   (b) Using separation of variables, find a solution candidate for \( \frac{dy}{dx} = y^{2/3} \) with the initial value \( y(a) = b \).
   (c) Using your answer above, find a solution to the IVP with \( y(0) = 0 \).
   (d) What is a second solution to the IVP with \( y(0) = 0 \)
   (e) Given that we have found two explicit solutions to the IVP with \( y(0) = 0 \), how many have a solution are for this IVP? Can you sketch some solution curves?
   (f) Find a solution to the IVP with \( y(0) = 8 \). Is this solution unique? If so, does the solution hold for all \( x \in \mathbb{R} \)?

3. Number of initial conditions needed
   (a) Find the general solution to the differential equation \( y'' = x^2 + 2 \).
   (b) Does the initial condition \( y(0) = 1 \) guarantee a unique solution?
   (c) How can we add to the initial condition to guarantee a unique solution?
   (d) How many initial conditions would you expect to need to get a unique solution to the differential equation \( y^{(n)} = f(x) \)?

4. Using Uniqueness to Derive Identities: This problem concerns the IVP \( \frac{dy}{dx} = 0; y(0) = 1 \)
   (a) Check that the IVP above has a unique solution.
   (b) Verify that \( y(x) = \cos^2 x + \sin^2 x \) is a solution.
   (c) Verify that \( y(x) = 1 \) is a solution.
   (d) What common identity does this imply?

5. Find general solutions for the following separable differential equations. Where initial conditions are given, find the particular solution.
   (a) \( y' = y \sin x \)            (c) \( y^2 y' = (y^4 + 1) \cos x \)
   (b) \( y' = 3x^2 y^2 + 3x^2; y(0) = 1 \)        (d) \( y' = y^2 + y - 6; y(0) = 1 \)

6. Consider the separable differential equation \( \frac{dy}{dx} = g(x)h(y) \). Let \( r \) be such that \( h(r) = 0 \).
   Explain why \( y(x) = r \) must be a constant solution of the equation.
7. Integrating factors

(a) State the product rule:  \( \frac{d}{dx}(f(x)g(x)) = \) ⬤

(b) Find functions for each of the ⬤ symbols in these examples of the product rule

i. \( \frac{d}{dx}(y(x) \cdot ⬤) = 2xy(x) + ⬤ \cdot y'(x) \)

ii. \( \frac{d}{dx}(y(x) \cdot ⬤) = \cos(x)y(x) + ⬤ \cdot y'(x) \)

iii. \( \frac{d}{dx}(y(x) \cdot ⬤) = e^x y(x) + xe^x y(x) + ⬤ \cdot y'(x) \)

(c) Each expression below can be multiplied by a function of \( x \) to make it look like the right-hand side of the product rule. Find that expression.

i. \( y' - y \tan(x) \)

ii. \( y' + \frac{y}{x} \)

(d) Use the above to solve the following differential equations.

i. \( y' - y \tan(x) = 1 \)

ii. \( y' + \frac{y}{x} - \ln(x) = 0 \)

8. Suppose your differential equation is of the form \( y' + P(x)y = Q(x) \). Verify that \( \rho = e^{\int P(x)dx} \) is ALWAYS an integrating factor to make the left side look like a product rule.

9. Solve the following differential equations. Remember \( e^{\int P(x)dx} \) might not be the integrating factor that makes your life easiest.

(a) \( y' + 2xy = x; y(0) = -2 \)

(b) \( 2xy' + y = 10\sqrt{x} \)

(c) \( y' - x(y + 1) = 1 + y \)

(d) \( y' + y \cot x = \cos x \)

10. Find solutions to the following differential equations. Be sure to look for singular solutions (solutions that don’t fit the general solution)

(a) \( y' = 6e^{2x-y}; y(0) = 0 \)

(b) \( y' = 2xy + e^{x^2} \)

(c) \( y' = 2xy + 3x^2e^{x^2}; y(0) = 5 \)

(d) \( y' = 2xy^2 + 3x^2y^2 \)

(e) \( (x^2 + 4)y' + 3xy = x; y(0) = 1 \)

(f) \( y' = \frac{1 + \sqrt{x}}{1 + \sqrt{y}} \)

An Ending Thought:  The essence of mathematics is not to make simple things complicated, but to make complicated things simple.

– S. Gudder