The Bar Construction

Let $k$ be a commutative ring and let $A$ be a $k$-algebra. Let $X$ be a right module over $A$ and let $Y$ be a left module over $A$. Then we can construct a simplicial $k$-module $\{B_n(X, A, Y)\}_{n=0}^{\infty}$ whose $n$th level is $X \otimes A^\otimes n \otimes Y$. The degeneracy maps insert a new copy of $A$ or $X$ with $A$, or $A$ with $Y$. We can turn this simplicial $k$-module into a chain complex $B(X, A, Y)$ of $k$-modules in the usual way, by taking the alternating sum of the face maps. This yields the bar complex, the usual chain complex for computing $\text{Tor}_A(X, Y)$.

Notice that this construction works in any monoidal category $C$, where $A$ is a monoid in that category, $X$ is a right module over $A$, $Y$ is a left module over $A$, and the final result $\{B_n(X, A, Y)\}_{n=0}^{\infty}$ is a simplicial object of $C$. If $C$ has a reasonable notion of geometric realization, then we can form an object $B(X, A, Y)$; this is the generalized bar construction.

Let’s consider the case where $C$ is the category of topological spaces. Let $G$ be a topological monoid, and choose $\ast$ and $\ast$ as our right and left $G$-modules. Then the above construction yields

$$B_n(\ast, G, \ast) = \ast \times G^n \times \ast = G^n$$

$$BG = \prod_{n=0}^{\infty} G^n \times \Delta^n / \left\{ \begin{array}{l}
(g_0, \ldots, g_{i-1}, 1, g_{i+1}, \ldots, g_{n+1}, t_0, \ldots, t_{n+1}) \\
= (g_0, \ldots, g_{i-1}, g_{i+1}, \ldots, g_n, t_0, \ldots, t_i + t_{i+1}, \ldots, t_{n+1}) \\
(g_0, g_1, \ldots, g_n, 0, t_0, \ldots, t_{n-1}) = (g_1, \ldots, g_n, t_0, \ldots, t_{n-1}) \\
= (g_0, \ldots, g_{i-1}, g_{i+1}, \ldots, g_n, t_0, \ldots, t_{n-1}) \\
(g_0, \ldots, g_{n-1}, g_n, t_0, \ldots, t_{n-1}, 0) = (g_0, \ldots, g_{n-1}, t_0, \ldots, t_{n-1})
\end{array} \right\}$$

The first equation comes from degeneracies, and the last three come from faces. Under the assumption that $G$ is discrete, we can give this a more geometric description. There is one $n$-simplex for each $n$-tuple of elements of $G$. If the $n$-tuple contains an identity element $1 \in G$, then the $n$-simplex collapses onto a simplex of lower dimension. So we can think of one $n$-simplex for each $n$-tuple of non-identity elements of $G$. Under this description, the $0$th face of the $(n + 1)$-simplex corresponding to $(g_0, g_1, \ldots, g_n)$ is the $n$-simplex corresponding to $(g_1, \ldots, g_n)$. The $i$th face is the $n$-simplex corresponding to $(g_0, \ldots, g_i g_{i+1}, \ldots, g_n)$, which if $g_i g_{i+1} = 1$ is further collapsed to the $(n - 1)$-simplex $(g_0, \ldots, g_{i-1}, g_{i+1}, \ldots, g_n)$ by mapping the $i$th and $i + 1$st vertices of the $n$-simplex to the $i$th vertex of the $(n - 1)$-simplex. Finally, the $(n+1)$st face of $(g_0, g_1, \ldots, g_n)$ is the $n$-simplex $(g_0, \ldots, g_{n-1})$.

Let $G \to H$ be a map of monoids. Then this clearly gives a map $G^n \times \Delta^n \to H^n \times \Delta^n$. It agrees with the face and degeneracy identifications because it preserves multiplications and the identity element; therefore we get a map $BG \to BH$. The identity map $G \to G$ yields the identity $BG \to BG$, and a composition of maps $G \to H \to K$ yields the composition $BG \to BH \to BK$, which can easily be checked by seeing that it works for the simplices themselves before we quotient anything down. So $B$ gives a functor from topological monoids to topological...
spaces. More generally, each map of triples \((X, G, Y)\) that preserves all the multiplications induces a map between their bar complexes.

More properties of \(BG\):

- \(BG\) always has a canonical basepoint \(G^0 \times \Delta^0\).
- If \(G\) is grouplike and has nondegenerate basepoint, there is a natural weak homotopy equivalence \(G \to \Omega BG\), given by the formula

\[
(g, t) \mapsto (g, t, 1 - t) \in G \times \Delta^1
\]

So we call \(BG\) a “delooping” of \(G\).
- \(B\) is a strong monoidal functor. In other words, \(B(G \times H)\) is naturally homeomorphic to \(BG \times BH\). This follows from Milnor’s Theorem \(|X \times Y| \cong |X| \times |Y|\). The homeomorphism is given on the simplicial spaces by

\[
(G \times H)^n \times \Delta^n \to (G^n \times \Delta^n) \times (H^n \times \Delta^n)
\]

by the projections onto each factor.
- If \(G\) is an abelian topological group, then multiplication \(G \times G \to G\) and inversion \(G \to G\) are homomorphisms. By the above, this implies that there is a multiplication map \(BG \times BG \cong B(G \times G) \to BG\) and an inversion map \(BG \to BG\) turning \(BG\) into a topological group. Therefore we can take \(B^2G = B(BG)\). In fact, we can drop the assumption that \(G\) has inverses. If \(G\) is just a commutative topological monoid, then we still have the multiplication map \(G \times G \to G\) and it turns \(BG\) into a topological monoid.
- If \(G\) is commutative, then \(BG\) is commutative as well. (Just check that the reverse of the above map on the diagonal in \(\Delta^n \times \Delta^n\) is commutative.) So we can take \(B^nG = B(B(\ldots (B(G)) \ldots))\) for any nonnegative integer \(n\). This turns any topological commutative group or monoid \(G\) into a topological commutative group or monoid \(B^nG\). This generalizes from commutative monoids to \(E^n\) spaces; we take \(B^n\) of an \(E^n\) space using a different construction found in loop space theory.
- If \(H\) acts on \(X\) on the left, and this commutes with \(G\) acting on the right, then \(H\) acts on \(B(X, G, Y)\) on the left. Same for the right-hand side.
- Define \(EG = B(\ast, G, G)\); then when \(G\) is a group, it acts freely on \(EG\) on the right, \(EG\) is contractible, and \(EG/G \cong BG\). So \(EG \to BG\) is a universal principal \(G\)-bundle. That is, if \(X\) is homotopy equivalent to a paracompact space, then there is a natural bijection between \([X, BG]\) and isomorphism classes of principal \(G\)-bundles over \(X\). (Recall that a principal \(G\)-bundle is a locally trivial fibration with a fiberwise right \(G\)-action giving homeomorphisms between \(G\) and each fiber; it could also be described as a fiber bundle with fiber \(G\) and structure group \(G\) acting on the left.)
• We can generalize the last bullet point to $G$ any grouplike monoid using [2]. In this case, $EG \to BG$ is only a quasifibration with a right $G$-action; we can apply the functor $\Gamma$ to replace it by an equivalent “$G\eta$-fibration.” Then over any space $X$ homotopy equivalent to a CW complex, $[X, BG]$ is in natural bijection with equivalence classes of “principal $G$-fibrations.” This last notion refers to maps $E \to X$ that are quasifibrations with a fiberwise right $G$-action giving weak equivalences $G \to E_x$ by $g \mapsto yg$ for any point $y \in E$ over $x \in X$. Equivalences are generated by the equivariant fiberwise maps. We can strengthen from quasifibrations to Serre fibrations, Hurewicz fibrations, or “$G\eta$ fibrations” and get the same result. If $G$ has the homotopy type of a CW complex, we can also restrict to spaces such that the maps $G \to E_x$ are strong homotopy equivalences.

• If $X$ is a space, then $X \cong B(X, *, *) \cong B(*, *, X)$. If $X$ and $Y$ are spaces, then $X \times Y \cong B(X, *, Y)$. If $X$ has a right $G$-action and $Y$ has a left $G$-action, then $B(X, G, Y)$ is a homotopy-theoretic version of $X \times_G Y$.

• For every inclusion of groups $H \hookrightarrow G$ (not necessarily normal) we can form quotient spaces of left cosets $G/H$ and right cosets $H \backslash G$. Then $G/H \cong B(G, H, *)$ and $H \backslash G \cong B(*, H, G)$.

• If $H \to G$ is any homomorphism, not necessarily an injection, than we can use the bar complexes $B(G, H, *)$ and $B(*, H, G)$ as the definition of the generalized homotopy quotients $G/H$ and $H \backslash G$. Then the two rows of this diagram are equivalent fibration sequences:

\[
\begin{array}{cccccc}
H & \to^f & G & \to & G/H & \to & BH & \to & BF & \to & BG \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\Omega BH & \to & \Omega BG & \to & F(Bf) & \to & BH & \to & BF & \to & BG \\
\end{array}
\]

• We can carry out a two-sided bar construction $B(X, G, Y)$ anytime we’re in a context where we have associative multiplications between copies of $G$, $X$, and $Y$. For example, if $G$ is a monad (a functor $\text{Top} \to \text{Top}$ that behaves like a monoid) on spaces, $Y$ is a space that is a left $G$-module, and $X$ is a functor that is a right $G$-module, we can define $B_n(X, G, Y)$ in a similar way and get a simplicial space. Taking $G$ to be the little $n$-cubes operad, $X = \Omega^n$, and $Y$ an algebra over the little $n$-cubes, this construction yields an $n$-fold delooping of $y$.

• We can also generalize the reduced bar construction $B(*, G, *)$ from topological monoids $G$ to topological categories. Instead of points of $G$, we consider morphisms in such a category. The $n$th space is defined as above, though we must require that each $n$-tuple of arrows is composable, i.e. the target of each arrow is the source of the next. In the case of a one-object category, this gives exactly the same construction as above.

References
