MATH 348 SECTION C1

MIDTERM 3 PRACTICE PROBLEMS

There are 5 problems, each worth 6 points, for 30 points total. You have 50 minutes. Please write clear, complete, and concise solutions. Do not skip any major logical steps, and clearly label the important steps of the proof, along with which technique or axiom you are using, if it has a name (e.g. Monotone Convergence Thm). Everything must be proven, unless the problem is marked (No proof). You may assume any results from class or from the book, unless otherwise explicitly noted.

As for what kind of help is allowed: No books, no notes, no calculators, no devices with a screen and/or an internet connection, and absolutely no messenger pigeons.

(1) (a) If \( a_n \) is bounded then \( a_n \) converges.
(b) If \( a_n \) has a convergent subsequence then \( a_n \) is bounded.
(c) If \( a_n \) is Cauchy then \( a_n \) is bounded.
(d) If \( a \in \mathbb{N} \) then \( 0 \mid a \).
(e) If \( a \mid n \) and \( b \mid n \) then \( ab \mid n \).
(f) If \( p \) is prime and \( a, b \in \mathbb{N} \) such that \( ab = p \), then \( a = p \) or \( b = p \).
(g) If \( p \) is prime then \( a^p \equiv 1 \mod p \).

(2) (a) State the Cauchy Convergence criterion.
(b) State the Bolzano-Weierstrass Theorem.

(3) Let \( a_n \) be a Cauchy sequence. Prove that \( a_n \) is bounded. \((Hint: \) Take \( \epsilon = 1 \) in the definition of Cauchy to get some \( N \in \mathbb{N} \). Prove that the terms are bounded by \(|a_N| + 1 \) and the maximum of \(|a_1|, \ldots, |a_{N-1}|\).\)

(4) (a) State the comparison test for series.
(b) Sketch an inductive proof that \( 3^n - 1 \geq 2^n \) when \( n \geq 1 \).
(c) Prove that \( \sum_{n=1}^{\infty} \frac{1}{3^n-1} \) converges.

(5) Prove that if \( a \) and \( b \) are integers then \( \gcd(a, b) = \gcd(a - b, b) \).

(6) (a) Compute the gcd of 78 and 90 using the Euclidean algorithm.
(b) Compute the gcd of 78 and 90 using prime factorization. \((Hint: \) \( 78 = 6 \cdot 13 \)).
(7)  (a) Solve $5x \equiv 3 \mod 24$. Express the answer either as an integer or an element of $\mathbb{Z}/24\mathbb{Z}$, but please make the final answer lie between 0 and 23. (Hint: $5 \cdot 5 = 24 + 1$.)

(b) Solve $5x \equiv 3 \mod 11$.

(8)  (a) State the Chinese Remainder Theorem in two different ways.

(b) Solve $5x \equiv 3 \mod 264$. (Hint: $264 = 11 \cdot 24$, so we can use the previous problem. Also notice that $24 \cdot 6 \equiv 2 \cdot 6 \equiv 1 \mod 11$.)

(9)  (a) State Fermat’s Little Theorem.

(b) Prove that if $p$ is prime and $k \equiv \ell \mod (p - 1)$ then $a^k \equiv a^\ell \mod p$.

(10)  (a) Compute $100,000 \mod 52$.

(b) Compute $2^{100,000} \mod 53$. 