Problem set.

(1) We will prove part (b) of Thm 13.25 stated in class. Let \( y \in [0,1) \) and suppose \( \ell_n \) is the largest multiple of \( \frac{1}{10^n} \) which is less than or equal to \( y \), as in class. We know that \( \ell_n = \ell_{n-1} + \frac{c_n}{10^n} \), where \( c_n \in \{0,1,\ldots,9\} \).

(a) Suppose that the digits \( c_n \) end with trailing 9's:
\[
c_N = 9, \quad c_{N+1} = 9, \quad c_{N+2} = 9, \quad \ldots
\]
In other words, there is some \( N \in \mathbb{N} \) such that \( c_n = 9 \) whenever \( n \geq N \).

Prove that for any \( n \geq N \),
\[
\ell_n = \ell_N + \frac{1}{10^N} - \frac{1}{10^n}
\]
(b) Continuing to assume that the decimal ends with trailing 9's, prove that \( y \geq \ell_N + \frac{1}{10^N} \). Explain why this is a contradiction.

(2) We will finish the proof of part (c) of Thm 13.25. Let \( \ell_1, \ell_2, \ldots \) be any sequence such that \( \ell_1 = \frac{c_1}{10} \), \( \ell_n = \ell_{n-1} + \frac{c_n}{10^n} \), and every \( c_n \) is in the set \( \{0,1,\ldots,9\} \). We proved in class that \( \ell_n \) converges to some \( y \in [0,1] \). Suppose that the \( c_n \) do not end in trailing 9's. Prove that the limit \( y \) must be smaller than 1.

(3) Let \( P(\mathbb{N}) \) denote the power set of \( \mathbb{N} \), the set of all subsets of \( \mathbb{N} \). Every element of \( P(\mathbb{N}) \) is a subset \( S \subset \mathbb{N} \). Adapt the “diagonal argument” from class to prove that \( P(\mathbb{N}) \) is uncountable.

(4) Do Exercise 14.14, which finishes the proof of Thm. 14.5.

(5) • Prove that the sequence \( a_n = \frac{1}{n^2} \) converges to zero when \( k \in \mathbb{N} \). (This is different from the result in class, which says \( \frac{1}{k} \to 0 \).)
• Use the “arithmetic of limits” we have developed to evaluate
\[
\lim_{n \to \infty} \frac{7n^3 - 63n^2 + 17n + 5}{2n^3 + 1}
\]

Revision. Please revise the writing assignment from HW6 and turn it in for a second grade.

Writing assignment. Write a complete solution to the following problem, along with a clear explanation of why your solution is correct. You are required to type the answer in LaTeX.
We will prove that Newton’s method for approximating square roots works. Let $a$ be a real number which is greater than 1. Define a sequence by $x_1 = a$, and after that $x_{n+1} = \frac{1}{2} \left( x_n + \frac{a}{x_n} \right)$. Prove that this sequence has a limit $L$, and that $L^2 = a$. You may not assume that square roots exist; the point is to use this to prove that $\sqrt{a}$ must exist (and to give an algorithm computing it).

*Hint:* If you can prove that $x_n^2 \geq a$ for all $n$, you can use it to show that $x_1, x_2, \ldots$ is decreasing.