# Differential Geometry: Notes on groups and actions

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A **topological group** is a Hausdorff topological space $G$ which is also a group so that the group operations are continuous. That is:

- **multiplication:** $m : G \times G \to G$
  given by $m(g, h) = gh$, and
- **inversion:** $i : G \to G$
  given by $i(g) = g^{-1}$

are continuous maps.

If $G$ is also a smooth manifold and $m$ and $i$ are smooth maps, then $G$ is called a **Lie group**.

A (left) action of a topological group on a topological space $X$ is a continuous map

$$G \times X \to X$$

denoted $(g, x) \mapsto g \cdot x$, with the property that for every $g, h \in G$ and $x \in X$, one has

- $g \cdot (h \cdot x) = (gh) \cdot x$
- $\mathbf{1} \cdot x = x$

Here $\mathbf{1}$ is the identity element of $G$.

If $M$ is a smooth manifold and $G$ is a Lie group, then we say that an action of $G$ on $M$ is smooth if the action map

$$G \times M \to M$$

is smooth.

Note that if $G$ acts on $X$ and $Y \subset X$ is *invariant* (meaning $g \cdot y \in Y$ for every $g \in G$ and $y \in Y$), then $G$ restricts to an action on $Y$.

Suppose $G$ acts on $X$. If, for any compact set $K \subset X$, the set

$$\{g \,|\, gK \cap K \neq \emptyset\}$$

is compact, then we say that the action of $G$ is **proper**.

We sometime refer to groups with the discrete topology as a **discrete groups**, and an action by a group with the discrete topology as a discontinuous action. Note that a proper and discontinuous (sometimes called, properly discontinuous) action of a group $G$ on $X$ has the property that for every $K$, the set defined above in $G$ is *finite*. An action is free if $g \cdot x = x$ implies $g = \mathbf{1}$.

Note that a countable discrete group is a Lie group.

The **orbit space** of $X$ by the action of a group $G$ is denoted $X/G$ and is the set of orbits of $X$ with the quotient topology.

$$\pi : X \to X/G$$