Math 520 Midterm: Take-home portion.

October 6, 2006

This is due Monday before the in-class portion of the exam begins. I would like you to work the problems on your own. You can use your notes and your books. But do not use the internet. Please write your solutions neatly and carefully. Good luck!

1. Let $\mathcal{A}_n$ denote the space of symmetric $n \times n$ matrices (it's clearly a smooth manifold of dimension $(n^2+n)/2$). Suppose
$$F : \mathbb{R}^n \to \mathcal{A}_n$$
is a smooth map, and $p \in \mathbb{R}^n$ is a point where $F(p)$ has distinct eigenvalues $a_1 < a_2 < ... < a_n$.

   Prove that the there exists a neighborhood U of p where the eigenvalues of $F(x^1, ..., x^n)$, for $(x^1, ..., x^n) \in U$, define n smooth functions on U.
   Hint: Consider the function $G(x^1, ..., x^n, t) = \det(F(x^1, ..., x^n) - tI)$.

2. Submanifolds $N, P \subset M$ are **transverse**, written $N \pitchfork P$ if for every $m \in N \cap P$
$$T_m N + T_m P = T_m M.$$
Equivalently, the embedding of P into M is transverse to N (or equivalently, the embedding of N into M is transverse to P). Here embedding means injective immersion for which the inclusion is a homeomorphism onto its image.

   Prove the following local transversality theorem:

   Theorem *If $N^n, P^r \subset M^k$ are embedded submanifolds and $N \pitchfork P$, then for every $m \in N \cap P$, there is a coordinate chart*
$$\phi : U \to (-1,1)^k$$
about m in M, so that
$$\phi(N \cap U) = \{(r^1, ..., r^n, 0, ..., 0) \,|\, r^j \in (-1,1)\}$$
$$\phi(P \cap U) = \{(0, ..., 0, r^{k-r+1}, ..., r^k) \,|\, r^j \in (-1,1)\}$$

3. Consider the following vector fields on $\mathbb{R}^4 \setminus \{0\}$:
$$\begin{aligned}
\xi_1 &= -x^2 \tfrac{\partial}{\partial x^1} + x^1 \tfrac{\partial}{\partial x^2} + x^4 \tfrac{\partial}{\partial x^1} - x^3 \tfrac{\partial}{\partial x^4} \\
\xi_2 &= -x^3 \tfrac{\partial}{\partial x^1} - x^4 \tfrac{\partial}{\partial x^2} + x^1 \tfrac{\partial}{\partial x^1} + x^2 \tfrac{\partial}{\partial x^4} \\
\xi_3 &= -x^4 \tfrac{\partial}{\partial x^1} + x^3 \tfrac{\partial}{\partial x^2} - x^2 \tfrac{\partial}{\partial x^3} + x^1 \tfrac{\partial}{\partial x^4}
\end{aligned}$$

   (a) Show that each ξ_i restricts to a nowhere vanishing, smooth vector field on S^3.
   (b) Show that at every $p \in S^3$, the tangent vectors $\{\xi_1(p), \xi_2(p), \xi_3(p)\}$ form a linearly independent set in $T_p(S^3)$.
   (c) Show that $TS^3 \cong S^3 \times \mathbb{R}^3$.