Some point-set topology

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Disclaimer

These are not complete notes on point set topology, but rather a list of definitions and properties that we will freely assume. If you are unfamiliar with the basics of point set topology, in particular, if the notions described below are foreign to you, I suggest consulting one of the standard texts (for example, Munkres' book "Topology; a first course"). It is worth noting that we will not use a great deal of point set topology, and the spaces we consider are pretty nice (as topological spaces go).

0.1 A topology

Given a set X, a collection $\mathcal{T}$ of subsets of X is called a **topology** on X if the following three conditions are satisfied:

1. $X \in \mathcal{T}$ and $\emptyset \in \mathcal{T}$.
2. If $\{U_1, ..., U_n\} \subset \mathcal{T}$ for some $n \in \mathbb{Z}_+$, then
$$\bigcap_{k=1}^{n} U_k \in \mathcal{T}$$
3. If $\{U_\alpha\}_{\alpha \in J} \subset \mathcal{T}$ for some index set J, then
$$\bigcup_{\alpha \in J} U_\alpha \in \mathcal{T}$$

A **topological space** $(X, \mathcal{T})$ is a set X together with a topology $\mathcal{T}$ on X. The elements of $\mathcal{T}$ are called the **open sets** of the topology $\mathcal{T}$. An open set containing a point $x \in X$ is called a **neighborhood** of x. We sometimes refer to X as a topological space with the topology being understood.

We can restate the definition of a topology as follows. In a topology, the whole set and the empty set are open. Furthermore, finite intersections of open sets are open and arbitrary unions of open sets are open.

The complement of an open set is called a **closed set**. Not surprisingly, a topology can alternatively be described by its closed sets: The whole set and the empty set are closed, arbitrary intersections of closed sets are closed, and finite unions of closed sets are closed.

Given a topological space $(X, \mathcal{T})$, and a subset $E \subset X$, the **closure** $\overline{E}$ of E is the intersection of all closed sets containing E. Said differently, $\overline{E}$ is the smallest (with respect to inclusion) closed set containing E. The **interior** of E, denoted E°, is the largest open set contained in E. It follows that $E^\circ = X \setminus \overline{(X \setminus E)}$.

Example 0.1. *Given a set X, there are two silly topologies on it. The first is called the **discrete topology** in which every set is open. That is, the topology is simply the power set on X.*

*The second is called the **indiscrete topology** in which only X and $\emptyset$ are open. We leave it as a trivial exercise to verify that these are indeed topologies on X. We will not have much use for them. More (familiar) examples will be given below.*

0.2 Bases

Given a set X, a **basis** for a topology on X is a collection $\mathcal{B}$ of subsets of X with the following properties.

1. If $x \in X$ then there exists $B \in \mathcal{B}$ such that $x \in B$.
2. If $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, then there exists $B_3 \in \mathcal{B}$ such that $x \in B_3$ and $B_3 \subset B_1 \cap B_2$.

If $\mathcal{B}$ is a basis for a topology on X, then the **topology generated by** $\mathcal{B}$ is the collection $\mathcal{T}$ of all unions of elements of $\mathcal{B}$. It is an exercise to see that this is indeed a topology. Note that $\mathcal{B} \subset \mathcal{T}$. We also say that $\mathcal{B}$ is **a basis for** $\mathcal{T}$.

Any topology $\mathcal{T}$ on X has a (silly) basis: simply take $\mathcal{B} = \mathcal{T}$.

Example 0.2. *The product of two topological spaces is a topological space. More precisely, if $(X_1, \mathcal{T}_1)$ and $(X_2, \mathcal{T}_2)$ are topological spaces, then the **product topology** on $X_1 \times X_2$ is the topology with basis $U_1 \times U_2$ where $U_1 \in \mathcal{T}_1$ and $U_2 \in \mathcal{T}_2$.*

Example 0.3. *If (X, d) is a metric space, then the set of balls is a basis for a topology on X:*

$$\mathcal{B} = \{B(x, r) \,|\, x \in X \; r > 0\}$$

*The is called the **metric topology***

If we take the standard euclidean metric on $\mathbb{R}^n$, then this makes $\mathbb{R}^n$ into a topological space. Unless otherwise stated, this is the topology we will put on $\mathbb{R}^n$ whenever we discuss it.

A topological space $(X, \mathcal{T})$ is called **second countable** if there exists a countable basis for $\mathcal{T}$.

Example 0.4. *$\mathbb{R}^n$ is second countable. Take, for example, the following for a basis*

$$\mathcal{B} = \{B(x, r) \,|\, x \in \mathbb{Q}^n \,,\, r > 0 \,,\, r \in \mathbb{Q}\}$$

0.3 Continuous functions

Suppose $(X_1, \mathcal{T}_1)$ and $(X_2, \mathcal{T}_2)$ are topological spaces. A function

$$f : X_1 \to X_2$$

is **continuous** if for every $U \in \mathcal{T}_2$, $f^{-1}(U) \in \mathcal{T}_1$.

It is an exercise to see that if $\mathcal{T}_1$ and $\mathcal{T}_2$ are topologies induced by metrics, then this agrees with the usual notion of continuous function between metric spaces.

A continuous bijective function with continuous inverse is called a **homeomorphism**.

The **support** of a continuous function $f : X \to \mathbb{R}$ is the closure of the set where f is nonzero:

$$\text{supp}(f) = \overline{\{x \in X \mid f(x) \neq 0\}}$$

0.4 Subspaces

If $(X, \mathcal{T})$ is a topological space, and $Y \subset X$ is *any* subset, then there is a topology on Y induced from $\mathcal{T}$ called the **subspace topology** on Y. We denote it by $\mathcal{T}_Y$, and it is defined by

$$\mathcal{T}_Y = \{U \cap Y \mid \text{ for every } U \in \mathcal{T}\}$$

It is an exercise to check that this does define a topology on Y.

It is another exercise to see that if $\mathcal{T}$ is induced by a metric on X, then the subspace topology on Y is induced by the metric of X restricted to Y.

0.5 Hausdorff property

A topological space $(X, \mathcal{T})$ is called **Hausdorff** if for any two points $x, y \in X$ with $x \neq y$, there exists open sets $U, V \in \mathcal{T}$ such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$. It is an exercise to check that metric spaces are Hausdorff.

0.6 Connectedness

A **separation** of a topological space X is a decomposition of X into two non-empty disjoint open sets. That is,

$$X = U \cup V$$

where U and V open sets, $U \neq \emptyset \neq V$, and $U \cap V = \emptyset$.

We say that X is **connected** if it has no separation.

We say that X is **path connected** if for any two points $x, y \in X$ there exists a path from x to y in X. That is, there exists a continuous function $\gamma : [0, 1] \to X$ (where $[0, 1]$ is the closed unit interval in $\mathbb{R}$) such that $\gamma(0) = x$ and $\gamma(1) = y$.

The image of a connected set by a continuous function is connected. It follows that a path connected space is connected.

A component (resp. path component) of X is a maximal connected (resp. path connected) subset. X is a disjoint union of its components (resp, path components).

0.7 Compactness

An **open cover** for a topological space $(X, \mathcal{T})$ is any collection of open sets $\mathcal{U} \subset \mathcal{T}$ such that

$$X = \bigcup_{U \in \mathcal{U}} U$$

A topological space $(X, \mathcal{T})$ is called **compact** if for any open cover $\mathcal{U}$ there is a finite subset $\{U_1, ..., U_n\} \subset \mathcal{U}$ such that

$$\bigcup_{k=1}^{n} U_k = X$$

That is, X is compact if every open cover has a finite subcover.

Example 0.5. *A subset $X \subset \mathbb{R}^n$ with the subspace topology is compact if and only if it is closed and bounded.*

In general, we refer to subsets of a topological space as compact if they are compact when given the subspace topology. If X is Hausdorff, then compact subsets are closed.

If $\mathcal{U}$ is an open cover of a topological space $(X, \mathcal{T})$, then an **open refinement** of $\mathcal{U}$ is another open cover $\mathcal{V}$ such that for all $V \in \mathcal{V}$ there exists $U \in \mathcal{U}$ such that $V \subset U$. Another way to phrase compactness is as follows: A topological space X is compact if and only if for every open cover there is a finite open refinement.

This motivates the following generalization of compactness. A cover $\mathcal{U}$ of X is called **locally finite** if for every point $x \in X$ there is a neighborhood V of x which meets only finitely many elements of $\mathcal{U}$.

A Hausdorff topological space X is called **paracompact** if every open cover has a locally finite refinement.

Another generalization of compactness is called local compactness. A topological space X is called **locally compact** if for every $x \in X$ there is an open set containing x which is contained in a compact subset of X.

...And yet another generalization of compactness is to say that a topological space X is **σ-compact**, which means that X is a countable union of compact sets.

0.8 Manifolds

An n**-dimensional (topological) manifold** or briefly an n**-manifold** is a second countable Hausdorff space X such that every point $x \in X$ has a neighborhood homeomorphic to an open set in $\mathbb{R}^n$. A manifold is an n-manifold for some n (this is not entirely standard: some people allow a manifold to have components of different dimension).

Example 0.6. $\mathbb{R}^n$ *is an* n*-manifold. Any open set in* $\mathbb{R}^n$ *is a manifold. Any open set in an* n*-manifold is an* n*-manifold. The product of two manifolds (with the product topology) is a manifold (the dimension is the sum of the dimensions).*

A manifold is connected if and only if it is path connected. Any component is a path component and vice versa. Moreover, the components are open sets, and are hence manifolds themselves.

A manifold is locally compact, σ-compact, and paracompact.

0.9 Quotient spaces

Let $(X, \mathcal{T})$ be a topological space and $\sim$ an equivalence relation on X. We define the **quotient space** of X by $\sim$ to be the topological space $(X/_\sim, \mathcal{T}/_\sim)$ defined as follows.

The set $X/_\sim$ is the set of equivalence classes X. There is a map

$$\pi : X \to X/_\sim$$

which sends a point of X to its equivalence class. We now declare a subset U of $X/_\sim$ to be open (i.e. to belong to $\mathcal{T}/_\sim$) if and only if $\pi^{-1}(U)$ is open in X (i.e. $\pi^{-1}(U) \in \mathcal{T}$).

Said differently, $\mathcal{T}/_\sim$ is the smallest topology for which π is continuous.

We will encounter this type of construction in a few different settings.