Differential Geometry: Notes on notation.

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For a vector space V, we have the following spaces associated to V:

- $$\mathcal{T}^r_s(V) = \mathcal{T}^0_s(V) \otimes \mathcal{T}^r_0(V)$$

  the homogenous tensors of type r, s. This is the s-fold tensor product of V with itself, tensor product with the r–fold tensor product of the dual space V^* with itself. So, these are linear combinations of the decomposable tensors of the form $v_1 \otimes \cdots \otimes v_s \otimes v^1 \otimes \cdots \otimes v^r$ where $v_1, ..., v_s \in V$ and $v^1, ..., v^r \in V^*$ are arbitrary vectors.

- $$\mathcal{T}(V) = \bigoplus_{r,s \geq 0} \mathcal{T}^r_s(V)$$

  here we define $\mathcal{T}^0_0(V) = \mathbb{R}$. This is an algebra with multiplication $\otimes$ defined on decomposable tensors by

  $$(v_1 \otimes \cdots \otimes v_s \otimes v^1 \otimes \cdots \otimes v^r) \otimes (u_1 \otimes \cdots \otimes u_{s'} \otimes u^1 \otimes \cdots \otimes u^{r'})$$

  $$= v_1 \otimes \cdots \otimes v_s \otimes u_1 \otimes \cdots \otimes u_{s'} \otimes v^1 \otimes \cdots \otimes v^r \otimes u^1 \otimes \cdots \otimes u^{r'}$$

- $$\mathcal{T}^*(V) = \bigoplus_{r \geq 0} \mathcal{T}^r_0(V) \text{ and } \mathcal{T}_*(V) = \bigoplus_{s \geq 0} \mathcal{T}^0_s(V)$$

  are two subalgebras of $\mathcal{T}(V)$. These algebras are graded.

- $$\mathcal{I}^*(V) < \mathcal{T}^*(V) \text{ and } \mathcal{I}_*(V) < \mathcal{T}_*(V)$$

  the two-sided ideals generated by the sets

  $$\{v^* \otimes v^* \,|\, v^* \in V^*\} \text{ and } \{v \otimes v \,|\, v \in V\}$$

  respectively.

- $$\Lambda^*(V) = \mathcal{T}^*(V)/\mathcal{I}^*(V) \text{ and } \Lambda_*(V) = \mathcal{T}_*(V)/\mathcal{I}_*(V)$$

  are the quotient algebras. Because $\mathcal{I}^*(V)$ and $\mathcal{I}_*(V)$ are graded ideals, the quotients retain their gradings.

  $$\Lambda^*(V) = \bigoplus_{r\geq 0} \Lambda^r(V) \text{ and } \Lambda_*(V) = \bigoplus_{s\geq 0} \Lambda_s(V)$$

  These direct sums are actually finite. The multiplication $\otimes$ on $\mathcal{T}(V)$ descends to a multiplication on $\Lambda^*(V)$ and $\Lambda_*(V)$ denoted $\wedge$.

- Linear maps between vector spaces often induce maps between these spaces that preserve the gradings.

  $$T : V \to W \quad \Rightarrow \quad \begin{array}{l} T^* : \mathcal{T}^*W \to \mathcal{T}^*V \\ T_* : \mathcal{T}_*V \to \mathcal{T}_*W \\ T^* : \Lambda^*W \to \Lambda^*V \\ T_* : \Lambda_*V \to \Lambda_*W \end{array}$$

For a smooth manifold M, we have several spaces associated to it.

- $$TM = \bigcup_{m\in M} T_mM \text{ and } T^*M = \bigcup_{m\in M} T^*_mM$$

  the tangent and cotangent bundles, made up of all the tangent and cotangent spaces.

- Applying the constructions above pointwise, we get many more smooth bundles over M. We mention here only the ones we are primarily interested in. There are others.

  $$\mathcal{T}^r_s(M) = \bigcup_{m\in M} \mathcal{T}^r_s(T_mM) \ , \quad \Lambda^*(M) = \bigcup_{m\in M} \Lambda^*(T_mM)$$

  and

  $$\Lambda^p(M) = \bigcup_{m\in M} \Lambda^p(T_mM)$$

- With all of these bundles, we have the associated vector spaces of sections (these are also modules over $C^\infty(M)$), we list those we are mostly interested in

  $$\begin{array}{rcllr} \mathfrak{X}(M) & = & \text{sections of } TM & : & \text{vector fields} \\ \Omega^*(M) & = & \text{sections of } \Lambda^*(M) & : & \text{differential forms} \\ \Omega^p(M) & = & \text{sections of } \Lambda^p(M) & : & \text{differential } p\text{–forms} \end{array}$$

  $\Omega^p(M)$ is naturally a subspace of $\Omega^*(M)$.

- A smooth map between manifolds $F : M \to N$ has a derivative dF_m at each point m. This induces a map on TM

$$dF : TM \to TN$$

given by $dF(m, \xi) = (F(m), dF_m(\xi))$.

- A smooth map also induces a map on Ω^*.

$$F^* : \Omega^*(N) \to \Omega^*(M)$$

given by

$$(F^*(\omega))_m = (dF_m)^*(\omega_{F(m)})$$

Here, $m \in M$ and $\omega \in \Omega^*(N)$. The subscript denotes the point of evaluation of the differential form. The same formula serves as a definition for the induced map on differential p–forms.

- There are two special subspaces of $\Omega^*(M)$: The *closed forms* $Z^*_{dR}(M)$, those forms ω with $d\omega = 0$ and the *exact forms* $B^*_{dR}(M)$, those forms that can be written as $d\omega$ for some $\omega \in \Omega^*(M)$. Because $d^2 = 0$, the quotient is a well-defined vector space $H^*(M) = Z^*_{dR}(M)/B^*_{dR}(M)$. This is the DeRham cohomology of M.