**Theorem.** If $\Gamma < \text{Isom}^+(\mathbb{H}^2)$ is a cocompact Fuchsian group, then

$$\mu \left( \mathbb{H}^2 / \Gamma \right) \geq \frac{\pi}{21}$$

**Proof.** Writing the signature $\text{sign}(\Gamma) = (g; c_1, \ldots, c_N)$, then the Gauss-Bonnet Theorem states:

$$\mu(\mathbb{H}^2 / \Gamma) = 2\pi \left( 2g - 2 + \sum_{i=1}^{N} \left( 1 - \frac{1}{c_i} \right) \right)$$

Note: $1 - \frac{1}{c_i} \geq \frac{1}{2}$ for each $i$ with equality if and only if $c_i = 2$.

Further, if $\text{sign}(\Gamma) = (g'; c_1', \ldots, c_N')$ and $g \leq g'$, $c_i \leq c_i'$ for each $i$, then

$$\mu(\mathbb{H}^2 / \Gamma) \leq \mu(\mathbb{H}^2 / \Gamma')$$

We now consider all possibilities for the signature. For each signature, we either bound the area from below by $\frac{\pi}{21}$ or else verify that there is no hyperbolic orbifold with that signature.

- **$g \geq 2$:**

  $$\mu(\mathbb{H}^2 / \Gamma) \geq 2\pi(2g - 2) \geq 4\pi$$

- **$g = 1$:**

  $$\mu(\mathbb{H}^2 / \Gamma) = 2\pi \left( \sum_{i=1}^{N} \left( 1 - \frac{1}{c_i} \right) \right)$$

  This is positive if and only if $N > 0$. In this case

  $$\mu(\mathbb{H}^2 / \Gamma) \geq 2\pi \frac{1}{2} = \pi$$

- **$g = 0$:**

  - **$N \geq 5$:**
    $$\mu(\mathbb{H}^2 / \Gamma) = 2\pi \left( -2 + \sum_{i=1}^{N} \left( 1 - \frac{1}{c_i} \right) \right) \geq 2\pi \left( -2 + \frac{5}{2} \right) = \pi$$
  
  - **$N = 4$:**
    $$\mu(\mathbb{H}^2 / \Gamma) = 2\pi \left( -2 + \sum_{i=1}^{4} \left( 1 - \frac{1}{c_i} \right) \right) = 2\pi \left( 2 - \sum_{i=1}^{4} \frac{1}{c_i} \right)$$

    This area is positive if and only if $(c_1, c_2, c_3, c_4) \neq (2, 2, 2, 2)$. Furthermore it is minimized when $(c_1, c_2, c_3, c_4) = (2, 2, 2, 3)$:

    $$\mu(\mathbb{H}^2 / \Gamma) = 2\pi \left( 2 - \left( \frac{3}{2} + \frac{1}{3} \right) \right) = \frac{\pi}{3}$$

  - **$N = 3$:**
    $$\mu(\mathbb{H}^2 / \Gamma) = 2\pi \left( 1 - \sum_{i=1}^{3} \frac{1}{c_i} \right)$$

    For this to be positive, we need

    $$\frac{1}{c_1} + \frac{1}{c_2} + \frac{1}{c_3} < 1$$

    This means $(c_1, c_2, c_3)$ can **not** be of one of the following types:

    $$(2, 2, n), n \geq 2; \quad (2, 3, n), n = 3, \ldots, 6; \quad (2, 4, 4); \quad (3, 3, 3)$$

    The minimal triples $(c_1, c_2, c_3)$ remaining, along with the area of their respective orbifolds are given by

    $$(2, 3, 7), \frac{\pi}{21}; \quad (2, 4, 5), \frac{\pi}{10}; \quad (3, 3, 4), \frac{\pi}{6}$$

  - **$N \leq 2$:**
    There are no hyperbolic orbifolds in this case.

This gives all the cases and so completes the proof.  \qed