The proof uses one version of the Frobenius Theorem...

Lecture 4

Def: A k-dimensional smooth distribution on a smooth n-manifold M, n ≥ k is a smooth subbundle of TM:

1. \( \mathcal{D} \subset TM \)
2. \( \mathcal{D}_m = T_m M \cap \mathcal{D} \subset T_m M \) is a k-dimensional vector subspace for all \( m \in M \)
3. For \( m \in M \), every neighborhood \( U \subset M \) of \( m \) and \( \xi_1, \ldots, \xi_k \in \mathcal{X}(U) \) such that \( \langle (\xi_i)_p, (\xi_j)_p \rangle = \delta_{ij} \) for all \( p \in U \).

Let \( T(\mathcal{D}) \subset \mathcal{X}(M) \) be the vector fields everywhere tangent to \( \mathcal{D} \) — \( \forall \xi \in T(\mathcal{D}), \xi \in T\mathcal{D} \forall m \in M \).

Ex: If \( \xi \in \mathcal{X}(M) \) is a vector field that is nowhere vanishing \( \xi \neq 0 \) for all \( m \in M \), then \( \mathcal{D} = \{ \xi \mid T\mathcal{D} \forall m \} \) is a 1-dimensional distribution on \( M \).

In this case, there is a 1-dimensional submanifold going through every point tangent to \( \mathcal{D} \) — integral curves.

In fact, there are local coordinates on \( M \), \( x = (x_1, \ldots, x_n) \), \( U \rightarrow \mathbb{R}^n \) such that \( \{x_2 = \text{const}_2, \ldots, x_n = \text{const}_n\} \subset U \) is tangent:

\[ x^1 \]
This purely local, so is true for any 1-diml distribution. 

**Definition.** Say a k-diml distribution \( D \) is **completely integrable** if about any point \( m \in M \) \( x = (x_1, \ldots, x_n) : U \to \mathbb{R}^n \) \( s.t. \)
\begin{align*}
N = \{ x_{k+1} = c_{k+1}, \ldots, x_n = c_n \} \subset U \quad \text{has} \quad &T_p N = T_p D \quad \forall \ p \in N, \\
&\forall (c_{k+1}, \ldots, c_n) \in \mathbb{R}^{n-k}
\end{align*}
i.e. \( N \) is tangent to \( D \) at every point. — such a manifold is called an integrable manifold.

If this is true, then observe that \( \forall \xi, \eta \in \Gamma(D) \), given \( x = (x_1, \ldots, x_n) : U \to \mathbb{R}^n \), \( N \subset U \) as above, we have \( \xi, \eta \in \mathfrak{X}(N) \) \( s.t. \)
\begin{align*}
\xi|_N &= \hat{\xi}, \\
\eta|_N &= \hat{\eta}
\end{align*}
— so if \( i : N \to M \)
the inclusion, \( \xi, \eta \) are \( i \)-related to \( \hat{\xi}, \hat{\eta} \). It follows that \( \langle \xi, \eta \rangle|_N = \langle \hat{\xi}, \hat{\eta} \rangle \in \mathfrak{X}(N) \) and \( \langle \xi, \eta \rangle \in \Gamma(D) \). That is, \( \Gamma(D) \) is a Lie subalgebra of \( \mathfrak{X}(M) \) — closed under \( [\cdot, \cdot]_N \).

**Converse is:**

**Frobenius Theorem:** \( D \) a k-diml distribution on \( M \), is completely integrable iff \( \Gamma(D) \) is a Lie subalgebra of \( \mathfrak{X}(M) \).

**Proof is inducin’ on dimension k — see Warner p. 42.**

[pick \( \xi \in \Gamma(D) \), flow preserves \( D \), project out flow direction ....]
There is also a differential form version:

**Theorem**: \( \mathcal{D} \) is completely integrable iff the annihilator

\[
\mathcal{K} = \{ \omega \in \Omega^k(M) \mid \omega|_\partial = 0 \}
\]

is closed under \( d \): \( d(\mathcal{K}) \subseteq \mathcal{K} \).

**Proof**: see Warner p.74.

\( \mathcal{K} = \mathcal{K}(\partial) \) is an ideal in \( \Omega^k(M) \), and if \( d(\mathcal{K}) \subseteq \mathcal{K} \),
it is called a differential ideal.

[So, \( \mathcal{D} \) is compl. integr. \( \iff \mathcal{K}(\partial) \) is a diff. ideal]

Suppose \( G \) is a Lie group. A Lie subgroup is a Lie group \( H \) and an injective homomorphism

\[
\phi : H \to G
\]

which is an immersion.

If \( h \) is the Lie algebra of \( H \), then \( \phi_* : h \to \mathfrak{g}_H \) is an injection, so we can view \( h \leq \mathfrak{g}_G \) as a Lie subalgebra.

Conversely, we have

**Theorem**: The connected Lie subgroups of \( G \) are in a 1-1 correspondence w/ Lie subalgebras \( h \leq \mathfrak{g} \).
idea of proof: Let $D_h$ be the distribution spanned by $h: (D_h)_g = \{g \xi | \xi \in \mathfrak{h}\}$. Then observe that for $X_i, Y_i \in \Gamma(D_h)$, we can write

$$X = \sum_i X_i \xi_i, \quad Y = \sum_i Y_i \xi_i,$$

where $\xi_1, \ldots, \xi_k$ is a basis.

and $X_i, Y_i \in \mathcal{C}^\infty(M) \forall i$. Then

$$[X, Y] = \sum_i \left( X_i [\xi_i, \xi_j] + X_i (g_i(Y_i)) \xi_j - Y_i (g_j(X_i)) X_i \right) e_i, (1.25)$$

so, $D_h$ is completely integrable, by Frobenius Theorem.

Can piece together integral manifolds to get a maximal integral manifold $\phi: M \rightarrow \mathcal{G}$ through any point.

Let $\phi: \mathcal{H} \rightarrow \mathcal{G}$ be the max. integral manifold through $e_i$. Can give the a Lie group structure at $e_i$ via homomorphism. See Warner p 94, for details. 13

Special case $\mathfrak{g} \subseteq \mathbb{R}^q$ 1-dimensional Lie subalgebra: $[\xi, \eta] = t_4 [\xi, \eta] = -t_5 [\xi, \eta] = 0$ (by skew-symmetry).

$\Rightarrow 1$-dimensional Lie subgroup ... How do we understand?

Let $\phi: \mathcal{G} \rightarrow \mathcal{G}$ be the local flow — so $\frac{d}{dt} \phi_t(g) = \xi_{\phi_t(g)}$, $t \in (-\epsilon, \epsilon)$.
Set \( \delta(t) = \varphi_t(e) \). Then, \( \forall g \in G \), since \( \delta \) is left mult:
\[
\frac{d}{dt} \delta(t) = \delta(t) \frac{d}{dt} \varphi(t) = \varphi(t) \varphi(t)(g)
\]
so \( \delta(t) \frac{d}{dt} \varphi(t) = \varphi(t) \varphi(t)(g) \) is also an integral curve.

\[
\Rightarrow \quad \delta(t) \frac{d}{dt} \varphi(t) = \varphi(t) \varphi(t)(g) \quad \Rightarrow \quad \varphi(t) = \text{right mult. by } h.
\]
\[\varphi(t) = \varphi_t(g),\]
\[\Rightarrow \quad \varphi_t(g) = \varphi_t(h)(g)\]
\[\Rightarrow \quad \varphi_t(g) = \varphi_t(h)(g)\]
\[\Rightarrow \quad \text{is defined } \forall t \in (-\infty, \infty), \text{map } \varphi_t(g)\]
\[\Rightarrow \quad \varphi_t(g) \text{ is defined } \forall t \in \mathbb{R}. \text{ Moreover, } \forall t, t' \in \mathbb{R}:\]
\[\delta(t + t') = \varphi_t(\delta(t)) = \varphi_t(\varphi(t')(g)) = \varphi(t')(\varphi(t')(g)),\] so:

\[\delta(t + t') = \varphi_t(\delta(t)) = \varphi_t(\varphi(t')(g)) = \varphi(t)(g)\]

\[\delta(t) = \delta(t)\]

is a homomorphism, and \( d\delta(t) = \delta \).

Any homomorphism \( \delta : \mathbb{R} \to G \) is called a \underline{1-parameter subgroup}.

The 1-parameter subgroups are in 1-1 correspondence with elements of \( G \). If \( \delta : \mathbb{R} \to G \) is the 1-par. subgroup of \( \delta(1) = e \), define \( \exp : \mathbb{R} \to G \) to be \( \exp = \varphi_t \).

Further set \( \exp : \mathbb{R} \to G \) to be given by \( \exp(\delta) = \exp(1) \).
Theorem: \( \forall \sigma, \mathcal{A}, \in \mathbb{R} \)

1. \( \exp(\sigma \mathcal{A}) = \exp(\sigma \mathcal{A}) \quad \text{--- exp is a hom. restricted to line} \)
2. \( \exp: \mathcal{A} \rightarrow \mathcal{G} \) is \( C^\infty \), \( \exp_0 \) is the identity, so \( \exp \) is a diffeo on a nbhd of \( \mathcal{O} \).

**Proof:** see Warner p105.

- Obviously have commutivity:
  \[
  \begin{array}{rcl}
  & G & \xrightarrow{\phi} H \\
  & \exp & \circ \quad \exp \\
  & a & \xrightarrow{\mathrm{d}\phi} h \\
  \Rightarrow & \phi \text{ determined by } \mathrm{d}\phi \text{ on connected } G.
  \end{array}
  \]

**Ex:** \( \exp: \mathcal{gl}_n \mathbb{C} \rightarrow \mathcal{GL}_n \mathbb{C} \)

- is given by matrix exponentiation:
  \[
  e^A = I + A + \frac{A^2}{2} + \cdots + \frac{A^j}{j!} + \cdots = \exp(A)
  \]
  (see Warner p105 for convergence)

- Can check:
  - \( B e^A B^{-1} = e^{BA} B^{-1} \)
  - \( \det(e^A) = e^{\text{tr}(A)} \).

**Ex** Lie algebra of \( \mathcal{SL}_n \mathbb{C} \):

\[
\mathfrak{t}_1(\mathcal{SL}_n \mathbb{C}) = \{ \mathcal{A} \in \mathcal{gl}_n \mathbb{C} / \det(e^A) = 1 \} = \{ \mathcal{A} \in \mathcal{gl}_n \mathbb{C} / \text{tr}(A) = 0 \} = \text{e}^{\text{tr}(A)}
\]
Defn. A (left) action $\varrho$ on a Lie group $G$ on $M$ is a smooth map $G \times M \to M$ written $(\sigma, m) \mapsto \varrho \cdot m = L_{\sigma}(m)$ satisfies:

1. $\varrho(\sigma_1 \sigma_2, m) = (\varrho(\sigma_2), \varrho(\sigma_1)) \cdot m$

2. $e \cdot m = m \quad \forall m \in M, \sigma, \tau \in G$

(a right action is defined similarly $M \times G \to M, R_{\sigma}$).

Note, $\forall \sigma \in G, L_{\sigma}^{-1} \colon M \to M$ is a diffeomorphism with inverse $L_{\sigma}^{-1} \cdot L_{\sigma} = 1$.

**Ex:** $GL_n(K) \times K^n \to K^n$

$(A, v) \mapsto Av$

- In standard coordinates, entries are polynomials.

**Ex:** $\phi : G \to GL_n(K)$ a Lie group homomorphism induces an action $G \times K^n \to K^n$

$(\sigma, v) \mapsto \phi(\sigma)v$.

**Ex:** Any Lie group acts on itself on the left and right:

$G \times G \to G$

$\sigma, r \mapsto \sigma r$

or $L_{\sigma} = l_{\sigma}, R_{\sigma} = r_{\sigma}$.

Aside: Any action $G \times M \to M$ induces a homomorphism $\phi : G \to \text{Diff}(M) = \{ f : M \to M \mid f \text{ a diffeomorphism} \}$

$\phi(\sigma) = L_{\sigma}$.

Can formally think of this as a Lie group homomorphism.
Q. What should the "Lie algebra" $\mathfrak{g}$ of $\text{Diff}(M)$ be?

A. $\mathfrak{X}(M)$. Note: There is an obvious map

$$ \text{d}\phi: \mathfrak{g} \to \mathfrak{X}(M) $$

with

$$ \text{d}\phi(g) = \left. \frac{d}{dt} \right|_{t=0} \exp(tg) $$

That is, 1-parameter subgroups of $G$ give rise to actions of $\mathbb{R}$ on $M$, i.e. flows. This is an antihomomorphism of Lie algebras:

$$ [\text{d}\phi(g), \text{d}\phi(h)] = -\text{d}\phi([g, h]) $$

$(M, g)$ a Riemannian manifold.

**Def:** An action of $G$ by isometries on $M$ is an action

$$ G \times M \to M $$

s.t. $L_g: M \to M$ is an isometry for all $g \in G$.

$$ G \to \text{Isom}(M, g) = \{ f: M \to M | f^*g = g \} $$

**Def:** A Riemannian manifold $(M, g)$ is homogeneous if

$\text{Isom}(M, g)$ is a Lie group acting transitively on $M$; $\forall m, p \in M \ \exists \ \sigma \in \text{Isom}(M, g) \text{ s.t. } \sigma^*m = p$. [In fact, $\text{Isom}(M, g)$ is always a Lie group]

More generally, a homogeneous space is a transitive action $\mathbb{G}$

a Lie group $\mathbb{G}$ on a manifold $G \times M \to M$

For $M$ connected, we will later prove

**Proposition** $\sigma_1, \sigma_2 \in \text{Isom}(M, g)$, w/ $\sigma_1(m) = \sigma_2(m)$, $d\sigma_1|_n = d\sigma_2|_n \Rightarrow \sigma_1 = \sigma_2$
Given $G \times M \rightarrow M$ an action, $m \in M$

Set $\text{Stab}_G(m) = \{ \sigma \in G \mid \sigma(m) = m \}$

Note $\text{Stab}_G(m) \times T_m M \rightarrow T_m M$ an action defined by

$(\tau, \nu) \mapsto d(L_\nu)_m(\tau)$

$\phi : \text{Stab}_G(m) \rightarrow \text{Aut}(T_m M)$ Lie group homomorphism.

Aside:  
An isomorphism $\eta : V \rightarrow \mathbb{R}^n$ defines an isomorphism of groups $\Phi : \text{Aut}(V) \rightarrow \text{GL}_n(\mathbb{R})$ given by $\Phi(A) = \eta A \eta^{-1}$

Given an inner product $B$ on $V$, we can choose $\eta$ to be an isometry $\eta : (V, B) \rightarrow (\mathbb{R}^n, \langle \cdot, \cdot \rangle)$

Then $\Phi^{-1}(O(m)) = O(n) = \{ A \in \text{Aut}(V) \mid A^*B = B \}$

$G \leq \text{Isom}(M, g)$, $\phi : \text{Stab}_G(m) \rightarrow O(g_m)$

is 1-1 by Proposition. Moreover, we have

\[ \Phi^{-1}(O(m)) = \{ A \in \text{Aut}(V) \mid A^*B = B \}. \]

Use Gram-Schmidt process to find an orthonormal basis for $(T_m M, g_m)$, let $\eta$ and this to $e_1, \ldots, e_n$.

Corollary: If $G < \text{Isom}(M, g)$, $G$ acts transitively on $M$ and

$\phi(\text{Stab}_G(m)) = O(g_m)$, then $G = \text{Isom}(M, g)$.

Proof: Given $G < \text{Isom}(M, g)$, let $\sigma \in G$ st. $\sigma(\sigma(m)) = m$. Then since $\text{Stab}_G(m) = O(g_m)$, $\exists \tau_2 \in G$ st. $\phi(\tau_2) = \phi(\sigma \sigma)$

$\Rightarrow \phi(\tau_2 \sigma \sigma) = \tau \Rightarrow 1 = 1 \Rightarrow \tau \sigma = e \Rightarrow \tau = \sigma \Rightarrow \tau \sigma_2 \in G$. \( \Box \)

Ex Euclidean space

$\mathbb{R}^n = (\mathbb{R}^n, g_{\mathbb{R}^n})$
$O(n) \times \mathbb{R}^n \to \mathbb{R}^n$ is by isometries, fixes $0$:

$$\text{Stab}_{O(n)}(0) = O(n)$$

$$\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$$

$$(v, x) \mapsto x + v$$

by isometries, acts transitively.

$\mathbb{R}^n, O(n) < \text{Isom}(\mathbb{R}^n)$. In fact these generate by Corollary

$$\text{Isom}(\mathbb{R}^n) = \left\{ x \mapsto Ax + v \mid A \in O(n), v \in \mathbb{R}^n \right\} \text{ transitive, } \text{Stab}_0(0) = \text{O}(n)$$

Can embed into $\text{GL}_n(\mathbb{R})$:

$$\text{Isom}(\mathbb{R}^n) \to \text{GL}_n(\mathbb{R})$$

$$(x \mapsto Ax + v) \mapsto (A \, | \, v)$$

$$\mathbb{R}^n \times O(n) = \text{Isom}(\mathbb{R}^n)$$

$$(v, A) \cdot (u, B) = (v + Au, AB)$$

Lie group & $\mathbb{R}^n$ is homogeneous

$L = \text{unit sphere}$

$$S^n = (S^n, g), \quad g = g_{S^n} \big|_{S^n}$$

$$O(n+1) \times \mathbb{R}^n \to \mathbb{R}^n$$

Given $A \in O(n+1)$, $\|Av\| = \|v\| \Rightarrow A(S^n) = S^n$. $A$ preserves $g_{S^n}$ & it preserves $g \Rightarrow O(n+1) < \text{Isom}(S^n)$

transitive — Gram-Schmidt again — and $\text{Stab}_{O(n+1)}(e_{n+1}) = O(n)$.

$$\text{Isom}(S^n) = O(n+1), \quad S^n \text{ is homogeneous } (O(n) \to O(n+1), A \mapsto (A|_i))$$

$$X_{n+1} = 1$$

Image preserves \{ $X_{n+1} = 1$ \}

$$(A|_i)(X) = (A + v|_i)$$

$\text{smooth}$$

Given $A \in O(n+1)$, $\|Av\| = \|v\| \Rightarrow A(S^n) = S^n$. $A$ preserves $g_{S^n}$ & it preserves $g \Rightarrow O(n+1) < \text{Isom}(S^n)$

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$$\text{Isom}(S^n) = O(n+1), \quad S^n \text{ is homogeneous } (O(n) \to O(n+1), A \mapsto (A|_i))$$

$$X_{n+1} = 1$$

Image preserves \{ $X_{n+1} = 1$ \}

$$(A|_i)(X) = (A + v|_i)$$

$\text{smooth}$
**Ex** Hyperbolic space: make analogy w/ $\mathbb{S}^n$

$\mathbb{R}^{n,1} = (\mathbb{R}^{n+1}, B_{n,1})$

$B_{n,1}(u,v) = u^T J_{n,1} v$

$L_{n,1} = \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right)$

$\alpha(u,1) < GL_{n+1}(\mathbb{R})$ preserving $B_{n,1} = \beta = \langle \cdot, \cdot \rangle$.

**Def** Given $v \in \mathbb{R}^{n,1}$, say it is

\[
\begin{cases}
\text{light-like} & \text{if } \langle v, v \rangle = 0 \\
\text{space-like} & \text{if } \langle v, v \rangle > 0 \\
\text{time-like} & \text{if } \langle v, v \rangle < 0
\end{cases}
\]

**Lecture 60**

**Proposition** $u = \left( \begin{array}{c} u_{10} \\ u \end{array} \right), v = \left( \begin{array}{c} v_{10} \\ v \end{array} \right) \in \mathbb{R}^{n,1}$

$u_{10} > 0$, $u_{10} v_{10} > 0$, neither space-like, then

$\langle u, v \rangle \leq 0$ w/ equality $\iff$ $u$ are multiples of each other.

**Proof:** wlog. $u_{10} = v_{10} = 1$, by scaling

$\langle u, v \rangle = \sum_{i=1}^{n} u_i v_i - u_{10} v_{10} \leq \sqrt{\sum_{i=1}^{n} u_i^2} \sqrt{\sum_{i=1}^{n} v_i^2} - 1 \leq \sqrt{\sum_{i=1}^{n} u_i^2} \leq 0$

$0 \geq \langle u, v \rangle = \sum_{i=1}^{n} u_i^2 - u_{10}^2 \Rightarrow \sqrt{\sum_{i=1}^{n} u_i^2} \leq 1$

check equality statement
The \( \langle , \rangle \)-hyperboloid is defined to be
\[ \mathcal{H}^n = \{ x \in \mathbb{R}^n \mid \langle x, x \rangle = -1 \} \] a hyperboloid of \( 2 \) sheets,
\[ \mathcal{H}^n = \mathcal{H}^n_+ \sqcup \mathcal{H}^n_- \quad \text{with} \quad \mathcal{H}^n_+ = \{ x \in \mathbb{R}^n \mid x_{n+1} > 0 \} \]
\[ \mathcal{H}^n_+ \cong \mathbb{R}^n \quad (x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_n, \pm \sqrt{1 + \sum x_i^2}) \]

\( O(n, 1) \) preserves \( \langle , \rangle \), so leaves \( \mathcal{H}^n \) fixed.

\( O(n, 1) \times \mathcal{H}^n \to \mathcal{H}^n \)

Let \( O^+(n, 1) \) be the index \( 2 \) subgroup fixing \( \mathcal{H}^n_+ \).

\[ T_x \mathcal{H}^n = \ker (d \langle , \rangle_x) = \ker (\sum_{i=1}^n dx_i x_i - x_n dx_{n+1}) \]
\[ = \{ v \in \mathbb{R}^n \mid \langle v, x \rangle = 0 \} = x^+ \quad \text{under} \langle , \rangle \]

By proposition, \( T_x \mathcal{H}^n \cap L_c = \delta_0 \delta_+ \Rightarrow g = \delta_+ \mid T_x \mathcal{H}^n \) is positive definite.

**Defn.** Hyperbolic \( n \)-space is

\[ \mathbb{H}^n = (\mathcal{H}^n_+, g) \]

\( O^+ (n, 1) < \text{Isom} (\mathbb{H}^n) \)

\( O(n) \to O^+(n, 1) \)

\( A \mapsto (A \circ_1) \rightarrow \text{Stab } (e_{n+1}) = O(n) \)

Transitive: "Gram–Schmidt" Given \( x \in \mathcal{H}^n_+ \)

pick an orthonormal basis \( x_1, \ldots, x_n \) for \( x^+ \), set \( x_{n+1} = x \) then

\[ A = (x_1 | \ldots | x_n) \in O^+(n, 1) \quad \text{and} \quad Ax_{n+1} = x. \]
\[ G \times G \rightarrow G \] left and right actions are both transition

if by isometries for a Riem. metric, then the metric is called \textit{left or right invariant}. If both, then \textit{bi-invariant}

left / right invariant metric \( g \) is determined by \( g_{\xi \eta} = g\big|_{T e G} \cdot \) connected

Proposition: A left invariant metric \( g \) on a Lie group \( G \) is bi-invariant

\[ g([\xi,\eta],\xi) + g(\eta,[\xi,\eta]) = 0 \quad \forall \xi,\eta \in g. \]

Sketch: \( \phi \in G \). \( \phi : G \rightarrow G \) is an isometry \( \Rightarrow \phi \circ \phi^{-1} : G \rightarrow G \) is an isometry.

Note: \( \phi \circ \phi^{-1} \) is an automorphism of \( G \).

\[ \Rightarrow d\phi : g \rightarrow g \] isomorphism w/ \[ d\phi(\xi)(\eta) = d(\phi_{\xi})(\eta) \quad \forall \xi,\eta \in g. \]

Since \( g \) is left invariant, can check

\( \phi \) is an isometry \( \Rightarrow d\phi : T e G \rightarrow T e G \) is an isometry.

a. defines an action

\[ G \times G \rightarrow G \]

\( (\xi, \eta) \mapsto \phi_{\xi}(\eta) = \phi \circ \eta \circ \phi^{-1} \)

\( \text{Stab}_G(e) = G \) \( \Rightarrow \) \text{induced homomorphism}

\[ \text{Ad} : G \rightarrow \text{Aut}(T e G) \quad \text{in fact via} \ T e G \cong g \]

This is a homomorphism

\[ \text{Ad} : G \rightarrow \text{Aut}(g) \]

\( \eta \mapsto \text{Ad}_\eta \)

[Observe \( \text{Ad}_\eta = d(\phi_\eta) \) is actually an automorphism of the Lie algebra \( g \)]

\( g \) is bi-invariant \( \Rightarrow \text{Ad}(g) = O(g) \)
This homomorphism induces a Lie algebra homomorphism
\[ d(\text{Ad}) = \text{ad} : \mathfrak{g} \to \text{End}(\mathfrak{g}) \] (Lie algebra \( \mathfrak{g} \) \( \cong \) Aut(\( \mathfrak{g} \))) — adjoint rep of \( \mathfrak{g} \).

\( G \) connected \( \implies \) \( \text{Ad}(G) \subset O(\mathfrak{g}) \implies \text{ad}(\mathfrak{g}) \subset \mathfrak{o}(\mathfrak{g}) \).

Writing \( \text{ad} : \mathfrak{g} \to \mathfrak{g} \), we have

**Lemma** \( \text{ad}_g(\eta) = [\eta, g] \).

**Proof** — see Warner p 115

\( \mathfrak{o}(\mathfrak{g}) = \mathfrak{g} \)-skew symmetric end \( \mathfrak{g} \) \( \Rightarrow \) \( \text{End}(\mathfrak{g}) \)

\[ \{ A \in \text{End}(\mathfrak{g}) | g(A(\eta), \xi) = g(A(\xi), \eta) = 0 \} \]

So, \( \mathfrak{g} \) is bi-invariant \( \iff \) \( \text{ad}(\mathfrak{g}) \subset \mathfrak{o}(\mathfrak{g}) \iff \mathfrak{g}(\eta, \xi) + \gamma(\mathfrak{g}, [\xi, \eta]) = 0 \)

\[ \forall \xi, \eta \in \mathfrak{g} \]

[\( Rk = \implies \) holds w/out conn. assumption]

One last basic piece of structure on \((M, g)\):

An oriented Riemannian manifold comes equipped with a **canonical** volume form \( \omega = \omega_g \) with the property

that if \( x_1, \ldots, x_n \in T_x M \) is an orthonormal basis, then

\[ \omega(x_1, \ldots, x_n) = 1 \]

\( g \) also defines a diffeomorphism \( g_* : TM \to TM \) by \( g_* (v_m) = g(v_m, \cdot) \)

— turns v.f's into 1-forms.
A frame field on $U \subset M$ is a set $\xi_1, \ldots, \xi_n \in \mathcal{X}(U)$ s.t.

$\forall m \in U \xi_{(m)} = \xi_m$ is a basis for $T_m U = T_m M$.

Given a frame field, Gram-Schmidt produces an orthonormal frame field $\eta_1, \ldots, \eta_n \in \mathcal{X}(U)$ — "G-S is smooth!"

Volume form is $\omega = \eta^1 \wedge \ldots \wedge \eta^n$ for any pair of frame fields $\xi_{1}, \xi_{n}$.

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Connections (Ch2 DoCarmo)

Motivation: line segments in $\mathbb{R}^n$ uniquely minimize length for paths between points. Can characterize these, among all constant speed paths, as $\gamma: [a,b] \to \mathbb{R}^n$ s.t. $\gamma'' = 0$.

$\gamma''$ is not defined on an arbitrary mfd $M$ (at least, not as a map to $M$ or $TM$), it is a map to $T(TM)$. Here we use the canonical isomorphism $\pi^*_m \mathbb{R}^n = T_m \mathbb{R}^n$. $\gamma$ unit speed $\Rightarrow |\gamma'(t)| = 1 \forall t$.

$\gamma''(t) = \frac{D}{dt}(\gamma'(t))$ — measures how $\gamma'(t)$ turns in $\mathbb{R}^n$.

$\gamma(t) = (r \cos(t), r \sin(t))$, $\gamma'(t) = (-\sin(t), \cos(t))$, $|\gamma'(t)| = 1$.

$\gamma''(t) = \frac{1}{r} (-\cos(t), -\sin(t))$

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Diagram:

- Sharp turn
- Not so sharp turn