Proposition B: Suppose \( G \) acts on \( M \) and \( N \) and \( f: M \rightarrow N \) is a proper equivariant surjective map. Then the action on \( M \) is properly discontinuous iff the action on \( N \) is.

Proof: Given \( K \subset N \) open, \( f^{-1}(K) \) is open. Furthermore, given \( g \in G \),

\[
g \cdot K \cap \emptyset \iff f^{-1}(g \cdot K) \cap \emptyset \iff f^{-1}(K) \cap f^{-1}(g^{-1}K) \cap \emptyset \iff g \cdot f^{-1}(K) \cap f^{-1}(K) \cap \emptyset
\]

\( \iff g \cdot f^{-1}(K) \cap f^{-1}(K) \neq \emptyset \)

1. \( G \subset N \) PD. \( \Rightarrow \) \( G \subset M \) PD.

Observe that \( V \subset K \subset M \) open, \( f^{-1}(V) \) is also open and

\[
\{ g^{-1} \mid g \cdot f^{-1}(V) \cap V \neq \emptyset \} \subset \{ g^{-1} \mid g \cdot f^{-1}(V) \cap f^{-1}(V) \neq \emptyset \}
\]

So, from above, get

\( G \subset M \) PD. \( \Rightarrow \) \( G \subset N \) PD. \( \Box \).

Consider \( Isom(M_{m\theta}) = G \), \( m \in M \) and define

\( F: G \rightarrow M \)

by \( F(\sigma) = \tau \cdot m \). Clearly \( F \) is \( G \)-equivariant.

Lemma \( F \) is a proper submersion.

Proof: \( M \) homogeneous \( \Rightarrow F \) is a submersion. Indeed,

Let \( m, m' \in M \) be regular values, \( m \neq m' \), any point, \( \sigma \in F^{-1}(m) \), \( \sigma' \in F^{-1}(m') \).

Rays equivalence: fix base

\[
dF_{\sigma}(dlog_{\sigma}^*) \circ dF_{\sigma'} = d(log_{\sigma'}^{-1} \circ dF_{\sigma})
\]

where \( L_{\sigma}: M \rightarrow M \) is given by \( L_{\sigma}(p) = \tau \cdot \sigma(p) \). Since \( RHS \) is ant, so is \( LHS \).

Recall \( Stab(m_{\sigma}) \) is a closed subgroup of \( O_{\eta}(m) \cong O_{n} \) and is therefore compact.

Note: \( F^{-1}(m) = \sigma \cdot Stab(m_{\sigma}) \) where \( \sigma \in F^{-1}(m) \) is any \( \sigma \).

\( = \log(Stab(m_{\sigma})) \)
Let \( \Sigma \subset G \) be a subgroup with \( \dim \Sigma + \dim \text{Stab}(\gamma) = \dim(G) \) containing \( \sigma \). Then \( dF_{\tau, \gamma, \Sigma} \) is surjective, and small \( \delta \) and \( U \) s.t. \( \forall \sigma \in \Sigma, F|_U : U \rightarrow V \subset M \) is a diffeo, where \( V = B_{\delta}(m) \) for some \( \delta > 0 \).

Consider the map \( G : U \times \text{Stab}(\gamma) \rightarrow G \)
\[ (\gamma, \rho) \rightarrow \gamma \rho \]

At every pt \((\tau, \rho)\), we can check \( dG_{(\tau, \rho)} \) is an \( \infty \). Restrict to smaller \( V \) as necessary, we see \( G \) is a diffeo. Let \( \overline{B} = F_\rho^{-1}(\overline{B}_{\delta/2}(m)) \) which is closed, then
\[ G = (\overline{B} \times \text{Stab}(\gamma)) = F^{-1}(\overline{B}_{\delta/2}(m)) \] is closed.

Any compact \( K \subset M \) is contained in a finite union \( \bigcup \overline{B}_{\delta/2}(m) \), so \( F^{-1}(K) \) is contained in finite union of \( \gamma \) orbits, so \( \text{cpt} \). \( \square \)

We also need

\underline{Lemma} G a Lie group, \( \Gamma \subset G \) a subgroup. \( \Gamma \)-action on \( G \) is proper disc iff \( \Gamma \) is discrete.

\textbf{Proof:} Fix any left metrics on \( G \). Then \( G \) is homogeneous, hence complete. Since \( \text{cpl} \) sets are exactly closed, bounded sets, it suffices to show \( \Gamma \) is discrete if \( \forall \tau > 0, \left\| g_{\Gamma}(\tau) - g_{\Gamma}(e) \right\| < \tau \) \( \forall \tau > 0 \).

But, we have
\[ \Gamma \text{ is discrete} \iff \left\| g_{\Gamma}(\tau) - g_{\Gamma}(e) \right\| < \tau \] \( \forall \tau > 0 \).

Prop. follows from these two lemmas.
Let $k \subseteq \mathbb{Q}$ be a number field, i.e., a finite extension of $\mathbb{Q}$.

Let $k = \mathbb{Q}(\sqrt{d})$ be a number field.

Given $k = \mathbb{Q}(\sqrt{d})$, consider the set of all Galois embeddings

$$\sigma_d : k \to \mathbb{C}.$$ 

Assume $\sigma_0 = \text{id}$. 

Let $\mathfrak{c} = \mathbb{Q}(\sqrt{2})$. Then $\mathfrak{c} = \{a + b\sqrt{2} : a, b \in \mathbb{Q}\}$.

Let $\sigma_1, \sigma_2 \in \text{Gal}(k, \mathbb{Q})$.

Example: $k = \mathbb{Q}(\sqrt{5})$, $\mathfrak{c} = \mathbb{Q}(\sqrt{5})$. Then $\sigma_1(a + b\sqrt{5}) = a + b\sqrt{5}$.

For $k = \mathbb{Q}(\sqrt{d})$, $\mathfrak{c} = \mathbb{Q}(\sqrt{d})$, $d = \prod_{i=1}^{n} (\sigma_i(a + b\sqrt{d}))$.

$k$ is totally real if $\sigma_i(k) \subseteq \mathbb{R}$ for all $i$.

Recall: $d_{1x_0} = x_0^2 - dx_1$ is a basis for $k/\mathfrak{c}$ if $\det(T(x_0)) = a$,

$$a \Delta(x) = \sum_{i=1}^{n} x_i \sigma_i(x_0) = 0.$$ 

Let $R_k \subseteq k$ be the ring of integers, i.e., the set of elements $k$ satisfying a monic integral polynomial.

Proposition: If $d_{1x_0} \in R_k$, then $k/\mathfrak{c}$ is a basis for $k$. 

Let $k = \mathbb{Q}(\sqrt{d})$. Then $R_k = \mathbb{Z}[\sqrt{d}] = \{a + b\sqrt{d} : a, b \in \mathbb{Z}\}$.

$\alpha = \sqrt{d}$ if $d \equiv 2,3 \pmod{4}$.

$\alpha = \frac{1}{2}$ if $d \equiv 1 \pmod{4}$.

$\alpha = 1$ if $d \equiv 4 \pmod{4}$.

$\alpha = 1$ if $d \equiv 1 \pmod{4}$.
Consider the homomorphism
\[ \sigma = (\sigma_1, \ldots, \sigma_d) : k \to \mathbb{R}^d, \quad \sigma(x) = (\sigma_1(x), \ldots, \sigma_d(x)). \]

If \( x_{i-1} - x_d \) is a basis of \( k/\alpha \), then \( \sigma(x_{i-1}) - \sigma(x_d) \) is a basis for \( \mathbb{R}^d \)
since \( \sigma_i(\alpha) \neq 0 \). Taking \( x_{i-1} - x_d \) is an \( \mathbb{Z} \)-basis, then see
\[ \sigma(R_k) = \mathbb{Z}\sigma(x_1) + \cdots + \mathbb{Z}\sigma(x_d) \cap \mathbb{R}^d \text{ is a lattice in } \mathbb{R}^d / \sigma(R_k) \]

\( \text{of type } \left( \begin{array}{c} \text{Special tori from } k \end{array} \right) \]

Exercise 1: \( k = \mathbb{Q}(\sqrt{d}), \) (1d) \( \mathbb{Z} \)-basis, then
\[ a + b \alpha \mapsto (a + b \alpha, \alpha - b \alpha) \]
\[ d = 2; \quad \alpha + \beta \mapsto (a + b \alpha, \alpha - b \beta) \]

Now suppose \( B \) is a bilinear form defined on \( k \) or \( R_k. \)
\[ B(u, v) = u^T \beta v \quad \text{where } \beta \in M_{n \times n}(k) \text{ symmetric.} \]

Define \( \beta_i^k \) to be given by
\[ B_i^k(u, v) = u^T \sigma_i(\beta) v \quad (\sigma_i(\beta))_{k} = (\sigma_i(\beta_k)) \]

Assume \( \beta_i^k \) has signature \((n-1, 1)\) and \( \beta_j^k \) has signature \((1, n-1)\) \( \forall j \geq 1. \)

Exercise 2: \( \beta = \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right) \) defined on \( R_k, \) \( k = \mathbb{Q}(\sqrt{d}) \) \( \begin{array}{c} \text{basis in } R_k: e_i - e_{i+1} \end{array} \)

Signatures
\( (n-1, 1) \)
\( (1, n-1) \)

Observe \( \beta(\sigma_i) = \left[ \sum_{\lambda \in M_{n \times n}(k)} | A \sigma_i^A \beta^A \sigma_i | \right] \)
\[ \beta(\sigma_i) = \left( \begin{array}{c} \alpha_{i-1, i} \end{array} \right) \quad \text{change basis.} \]
The homomorphism \( \sigma \) induces a homomorphism

\[
\sigma : \bigoplus_{i=1}^{d} M_{n \times n}(k) \rightarrow \prod_{i=1}^{d} M_{n \times n}(K_i) \cong \prod_{i=1}^{d} M_{n \times n}(K_i) 
\]

Note: \( \sigma \) is discrete as \( \prod_{i=1}^{d} M_{n \times n}(K_i) \) is not a group.

Restricts to \( O(\mathbb{B}_i, \mathbb{R}_i) \) we have

\[
\sigma_i: O(\mathbb{B}_i, \mathbb{R}_i) \rightarrow \prod_{i=1}^{d} O(\mathbb{B}_i, \mathbb{R}_i) = \prod_{i=1}^{d} O(\mathbb{B}_i, \mathbb{R}_i) \cong O(n-1,1) \times O(n) \times \cdots \times O(n)
\]

1. Image of \( O(\mathbb{B}_i, \mathbb{R}_i) \) is discrete \( \Rightarrow \) action on \( O(n-1,1) \times \Pi \Omega \mathbb{A} \) is p.d.

2. \( \Pi \Omega \mathbb{A} \) is compact, so \( \sigma_i(O(\mathbb{B}_i, \mathbb{R}_i)) \) also acts p.d. on \( O(n-1,1) \) by Proposition B

3. Image of \( O(\mathbb{B}_i, \mathbb{R}_i) \) in \( O(n-1,1) \) is discrete.

With a little more work, we can find \( \Gamma < O(\mathbb{B}_i, \mathbb{R}_i) \) finite index in \( \Gamma \).

\( \text{image lies in } O^+(n-1,1) \& \text{ acts freely.} \)

\( \text{With a little more work one can show } \text{vol}(\Gamma^+_{\mathbb{R}}) < \infty. \)
(Locally) Symmetric spaces. (see Helgason: D.G., Lie gps, symm. spcs)

Recall from last hw: \((M,g)\) is locally symmetric if \(\mathcal{D}R = 0\) when \(R = \text{Ric} \, \text{curv}.

Eq (homoth.) \(M\) constant curvature \(\Rightarrow\) \(M\) locally symmetric (also follow from next def.)

Before we describe geometric way to think of locally symmetric, we need to fill in

Them A. Cartan, that in sense, implies that the "curvature determines the metric"

Let \((M,g)\) be Riemannian and \(m, \mathbb{R}^n, \mathbb{R}^p\). Consider the following map of normal balls:

\[ F : B_p(m) \to B_q(n) \]

\[ F = \exp_p \circ \iota \circ \exp^{-1} \quad \text{when} \quad \iota : T_m M \to T_{p}(\mathbb{R}^n) \quad \text{is a linear isometry} \]

\[ \forall \ p \in B_p(m), \ \text{let} \ \Phi_p : T_m M \to T_p(\mathbb{R}^n) \quad \text{be given by} \]

\[ \Phi_p = \tilde{P}_p \circ \iota \circ P^{-1}_p \quad \text{where} \quad P_p : T_m M \to T_p M \quad \text{is parallel transport along} \]

the radial geodesics \( \gamma : [0,\infty) \to B_p(m), \ \gamma(0) = m, \gamma'(0) = p. \) Similarly define \( \tilde{P}_p : T_p(\mathbb{R}^n) \to T_p(\mathbb{R}^n) \)

Then if \( \forall \ p \in B_p(m), \ \Phi_p \in T_p M \) on have

\[ g(R(\Phi_p(s), \Phi_p(t), \Phi_p(\xi))) = g^p(\Phi_p(s), \Phi_p(t), \Phi_p(\xi), \Phi_p(\xi)) \]

then \( F \) is an isometry with \( dF = \Phi_p \) \( \forall p \in B_p(m). \)
Let \( \gamma : (0,1) \rightarrow P_g(\mathcal{M}) \). We get for any \( p, \xi_1, ..., \xi_n \in \mathcal{X}(\gamma) \) parallel at some \( a \in (0,1) \). Let \( T \in \mathcal{X}(\gamma) \) be a Jacobian field with

\[
J(t) = \sum_{i=1}^n \xi_i(t) \xi_i(t)
\]

**Jacobi equation:**

\[
\sum_i \dot{\xi}_i(t) \Gamma^k_{ij}(t, \xi_i(t), T) \xi_j(t) = 0
\]

\[
\dot{w}(t) \xi_j(t) = \dot{\xi}_j(t) + g(\Gamma^k_{ij}, \xi_i(t), \xi_j(t), \xi_k(t)) = 0
\]

Set \( \tilde{\xi}_j(t) \in \mathcal{X}(\gamma) \), \( \tilde{\xi}_i(t) = \phi_{\gamma(a)}(\xi_i(t)) \). By assumption \( \tilde{\xi}_j(t) \) parallel, \( \tilde{\xi}_j(a) \equiv \tilde{\xi}_j(a) \)

Let \( T(t) \in \mathcal{X}(\gamma) \) be given by

\[
T(t) = \phi_{\gamma(a)}(J(t)) = \sum_i \xi_i(t) \xi_i(t)
\]

Then

\[
\frac{\partial^2 T}{\partial t^2} + \Gamma^k_{ij} \nabla_i T \nabla_j = \dot{\xi}_j(t) + g(\Gamma^k_{ij}, \xi_i(t), \xi_j(t), \xi_k(t)) = \dot{\xi}_j(t) + g(\Gamma^k_{ij}, \xi_i(t), \xi_j(t), \xi_k(t))
\]

So \( T \) is a Jacobi field.

If \( J(0) = 0 \) \& \( \tilde{T}(0) = 0 \) \& \( \tilde{T}(\ell) = d(\exp_p(\gamma(\ell)), \tilde{T}(0)) = \ell J_{\tilde{\gamma}}(\ell) \)

Similarly \( \tilde{T}(\ell) = d(\exp_p(\gamma(\ell)), \tilde{T}(0)) \). By the chain rule

\[
dF_p(J(\ell)) = \tilde{T}(\ell)
\]

And hence \( \Phi_{\tilde{\gamma}}(\tilde{\gamma}(\ell)) = \ell J_{\tilde{\gamma}}(\ell) \). \( \ell \) \& \( \tilde{\gamma}(\ell) \)

Any vector \( T \in TM \) appears as \( \tilde{\gamma}(\ell) \), so \( T \) is an element of \( \tilde{\gamma}(\ell) \)

Gives another alternative proof that \( J \) must come from parallelism of curvilinear coordinate.

 locally isometric.
We'll use this to prove the following:

\[ (M, g) \text{ Riem. m+M}, \text{ the local symmetry of } m \text{ is the diffeo.} \]

\[ \sigma : \mathcal{B}_m(M) \rightarrow \mathcal{B}_m(M) \text{ check m normal ball given by} \]

\[ \sigma(p) = \exp_m(-\exp_m(p)) \]

Then \( M \) is locally symmetrical \( \iff \forall p \in M, \text{ the local symmetry is the identity} \).

**Proof:** Suppose first \( M \) locally symmetrical, so \( \Delta \sigma = 0 \).

Pick an orthonormal basis of \( M \) at \( m \) and let \( \xi_j, j = 1, \ldots, n \in \mathcal{C}(\mathcal{B}_m(M)) \) be the coordinates parallel along radial geodesics, so setting

\[ \mathbf{R}_{ijkl} = g(R(\xi_i, \xi_j) \xi_k, \xi_l) \]

But, for any radial geodesics \( \xi \),

\[ \frac{d}{dt}(\mathbf{R}_{ijkl}(\xi(t))) = \mathbf{X}(t) \cdot g(R(\xi_i, \xi_j) \xi_k, \xi_l) = g(R(\xi_i, \xi_j) \xi_k, \xi_l) + g(R(\xi_i, \xi_j) \xi_k, \xi_l) + g(R(\xi_i, \xi_j) \xi_k, \xi_l) = 0 + 0 = 0 \]

\[ \mathbf{R}_{ijkl} \text{ is constant along radial lines} \]

If we let \( \phi_p : T_pM \rightarrow \mathcal{T}_pM \) be defined as above, the clearly

\[ \phi_p(\xi_j(p)) = -\xi_j(\sigma(p)) \]

So,

\[ g(R(\xi_i, \xi_j) \xi_k, \xi_l)(p) = \mathbf{R}_{ijkl}(p) = \mathbf{R}_{ijkl}(\sigma(p)) = g(R(\xi_i, \xi_j) \xi_k, \xi_l)(\sigma(p)) \]

\[ = g(R(-\xi_i, -\xi_j) \xi_k, -\xi_l)(\sigma(p)) = g(R(\sigma(\xi_i), \sigma(\xi_j)) \sigma(\xi_k), \sigma(\xi_l)) \]

Certain \( \sigma \) \( \iff \sigma \) is an isometry.
For the converse, fix \( m \in M \) and extend a parallel our frame \( \xi \) in \( M \) so that

\[ \xi(0) = m, \xi'(0) = \xi, \xi''(0) = X, \xi_3 = \xi \in \mathfrak{X}(M) \]

\[ \text{Reg} \xi(t) = g(\xi(t), \xi(t)) \in \mathfrak{X}(M) \]

\[ \frac{d^2 \xi}{dt^2} = \frac{\text{Reg} \xi(t) - \text{Reg} \xi(t+\varepsilon_0)}{2\varepsilon_0} \]

Note: isometric pure parallel \( \Rightarrow \xi(t) = \text{Reg} \xi(t) \)

and pure curve

\[ \text{Reg} \xi(-t) = g(\xi(-t), \xi(-t)) \]

\[ = g(\xi(\text{Reg} \xi(t)), \xi(\text{Reg} \xi(t))) = \text{Reg} \xi(t) \]

Thus, \( \nabla R = 0 \)

Converse: \( \text{curv} \equiv \text{sym} \equiv \text{locally sym} \equiv \text{globally sym} \)

Suppose \((M,g)\) is complete simply connected locally symmetric space of non-positive curvature, then Hadamard Theorem

\[ \Rightarrow \exp : T_M M \to M \text{ is a diffeo. Previous proof shows} \]

\[ \forall m \in M \text{ (locally symmetric) can be stood over } \]

and is an isometry, (globally) symmetric space

It follows that \( M \) is homogeneous: every \( m \in M \) by a geodesic

\[ \text{(Hopf-Rinow)} \]

\( \Rightarrow (M,g) \equiv M \quad \Rightarrow \text{from } M \text{ acts transitively.} \]
\[(M, g) \text{ sym. span}\]
\[\mathcal{G} = \text{Isom}_0(M) \quad (\text{compt exact } R)\]

have the map \(F: \mathcal{G} \to M\) such that \(\mathcal{G} = \text{Stab}_G(m_0)\)

Let \(\tau = \text{match}\) for \(\mathcal{G}\) and \(K\)

\[\tau: C \to C\]

\[\tau(K) = \text{the } \tau\text{-comm} \quad \forall K \Rightarrow \tau(K) = K\]

Let subalgs of \(\mathcal{G}\) associate to \(K\), \(t < q\) is exact

\[t = \{ \varepsilon \in \mathcal{G} \mid \tau(\varepsilon) = \varepsilon \} \]

\[\mathcal{A} = \{ \varepsilon \in \mathcal{G} \mid \tau(\varepsilon) = -\varepsilon \}\]

Hold \(dF \|_p : \mathcal{P} \xrightarrow{\cong} T_{m_0} M\)

(similar discussion)

Given \(\varepsilon \in \mathcal{A}\), let \(\varepsilon: \mathcal{P} \to M\) be the graph \(w/ \varepsilon(0) = m\) \(w/ \varepsilon(\alpha) = dF(\varepsilon)\)

\(\forall t \in \mathcal{P}, \text{ let } T_t: M \to M\) be the composites:

\[T_t(p) = \tau(\varepsilon(t)) \cdot t\]

\(\forall t \in \mathcal{P}, \quad T_t\tau(\varepsilon(t)) = \varepsilon(t) = T_t\varepsilon(0) = \varepsilon(0)\)

Consider parallel

\[d(T_t)_{(\varepsilon(0))}: T_{\varepsilon(0)}(M) \to T_{\varepsilon(0)}(M)\]

is parallel to \(t\) \(\Rightarrow t \to T_t\) hom \(\mathcal{P} \to G\)

\[\Rightarrow T_t(p) = \exp(t\varepsilon) \cdot p\]

\[\exp: \mathcal{P} \to G\]

So, \(\varepsilon \to \exp(t\varepsilon)p\) is good through \(p\)

- Can also compute converse for \(g\) in terms of \([t]_1\) on \(g\)
- Lots more structure!