So, for manifolds with positive sectional curvature \( K(p) \geq K_0 > 0 \), sufficiently long geodesics (at least \( \sqrt{K_0} \)) are not locally minimizing. \( \lambda \) [We'll see that Ricci curvature bounds, in fact, suffice].

In order to make this very useful, need to restrict to manifolds for which any two points can be connected by a geodesic.

**Lecture 19**

Hopf-Rinow & Harnolm theorem Ch. 7 B.C.

To use curvature hypotheses to study a Riemannian manifold, we must impose some further assumptions. If \((M, g)\) is Riemannian manifold, any curvature assumption on \( M \) are automatically valid for \( M - A \), where \( A \subset M \) is any closed subset.

For example, assuming \( M \) is compact, connected, eliminates this type of phenomena.

This is too restrictive for many applications — e.g., the universal covering of a compact manifold need not be compact.

A weaker hypothesis, completeness, enjoys many of the same properties, and is applicable in more situations (as we'll see).

Assume from now on (unless otherwise stated), manifolds are connected.
(Mg)-connected Riem. Mfd. is geodesically complete

if \( \forall m \in M, \exp_m \) is defined on all \( T_m M \).

**Theorem (Hopf-Rinow) Mg)-Riem. mfd. \( m \in M \). The following are equivalent:

(a) \( \exp_m \) is defined \( \forall m \in M \).

(b) \( K \in M \) is compact iff \( K \) is closed and bounded.

(c) \( M \) is complete.

(d) \( M \) is geodesically complete.

(e) \( M = \bigcup_{i=1}^{n} K_j \) w/ \( K_j \subset K_{j+1} \) opn cuts w/ \( N_j \), \( d(m, M - K_j) \rightarrow 0 \) as \( j \rightarrow \infty \).

Furthermore, any of the above imply

(f) \( \forall p \in M, \exists \text{ geodesic } \gamma : [0, \alpha] \rightarrow M, \gamma(0) = m, \gamma(\alpha) = p, d(m, p) = \ell(\gamma) \).

**Proof:** First we prove \((a) \implies (f)\).

What direction should we shoot off a geodesic from \( m \) to hit \( p \)?

Let \( \overline{B}_e(m) \) be a closed normal ball about \( m \), \( q_0 \in \overline{B}_e(m) \) closest to \( p \) — i.e., \( d(q_0, p) \leq d(q, p) \) \( \forall q \in \overline{B}_e(m) \). Clearly, either \( p \in \overline{B}_e(m) \) and \( p = q_0 \) and we're done, or else \( q_0 \in \partial B_e(m) = \text{normal sphere} \).

Then \( q_0 = \exp_m (\lambda v) \) for some \( \lambda \in T_m M, \lambda v = 1, \delta \). Set \( r = d(m, p) \).

Claim: \( p = \exp_m (r v) \). — \( v \) is the direction \( m \) should use.

Note, this is true iff

\[ \Delta = \{ t \in [0, r] \mid d(\gamma(t), p) = r - t \} = [0, r] \]

where \( \gamma(t) = \exp_m (tv) \).
We show that $\Delta$ is open & closed, and since $0 \in \Delta$ and $[0,1]$ is connected, we will have $\Delta = [0,1]$. Clearly $\Delta$ is closed since 

$\rho \mapsto (d(\xi(t), \rho) - r + t)$

is continuous and $\Delta$ is preimage of $0$. Suppose $t_0 \in \Delta$, $t_0 < r$ (if $t_0 = r$, then done). Let $\overline{B}_\delta(\xi(t_0))$ be a normal ball about $\xi(t_0)$, $\delta < r - t_0$.

Let $x_0 \in \overline{B}_\delta(\xi(t_0))$ be a closest point to $p$ in $\overline{B}_\delta(\xi(t_0))$.

Note: $d(\xi(t_0), x_0) + d(x_0, p) = d(\xi(t_0), p)$. Any path from $\xi(t_0)$ to $p$ can be replaced by one with length no longer for which the unit segment is a radial geodesic from $\xi(t_0)$ to $\overline{B}_\delta(\xi(t_0))$.

1. $r = d(m, p) \leq d(m, x_0) + d(x_0, p) \leq d(m, \xi(t_0)) + d(\xi(t_0), x_0) + d(x_0, p)$

   $= t_0 + \delta + r - t_0 - \delta = r$

So $\Delta$ is open. $\Delta$ is open.
(a) \rightarrow (b) \quad K \subset M \text{ closed & bounded} \\
\Rightarrow K \subset \exp_m(\overline{B}_r(0)) = \overline{B}_r(m) \quad \text{(by (f))}

\text{Convex is true in any metric space} \quad \checkmark

(b) \Rightarrow (c) \quad \text{Any Cauchy sequence is bounded and its closure is closed} \\
\Rightarrow \text{the Cauchy seq. has a convergent subseqnce} \\
\text{(c) \Rightarrow (d) Need to show that geodesics are defined at time t.}

\text{Suppose not: } \gamma : (a, b) \rightarrow M, \text{ geodesic, } \forall \gamma, \text{ sup \ max. speed} \not\Rightarrow \text{Suppose b < \infty.}

\text{Note } \left(\gamma\left(b-\frac{1}{n}\right)\right)_{n=1}^{\infty} \text{ is a Cauchy seq. and so converges to some m.}

\text{Let } W \text{ be an totally normal nbhd. of m.}

\gamma\left(\left[b-\frac{1}{n}, b\right]\right) \subset W. \text{ If } \frac{1}{n} < \delta, \text{ then the geodesic } \gamma|_{\left[b-\frac{1}{n}, b\right]}
\text{ can be extended to } \gamma|_{\left[b-\frac{1}{n}, b-\delta\right]} \quad \& \quad \text{\& so } \gamma \text{ can be extended to }
\gamma|_{\left[b-\frac{1}{n}, b-\frac{1}{n}+\delta\right]} \quad \Rightarrow \gamma|_{\left[a, b-\frac{1}{n}+\delta\right]} \supseteq (a, b) \forall n

(d) \Rightarrow (a) obvious from defn.

(b) \Rightarrow (c) true for any metric space. \quad \Box

\text{Corollary: } M \text{ compact } \Rightarrow M \text{ complete.}

\text{Corollary: } \hat{M} \rightarrow M \text{ a connected covering space. Then } M \text{ is complete} \Rightarrow \hat{M} \text{ is complete.}

\text{Proof: see homework 15.}

\text{EX: } \mathbb{R}^n, \mathbb{H}^n, S^n \text{ are complete since they are homogeneous - see hack as are any quotient by isometry groups acting freely and p.p.}

\text{EX: } G \text{ a Lie group w. a left (right) mvt metric - also homogeneous}
Theorem (Hadamard). Let \((M^n, g)\) be a complete Riemannian manifold with sectional curv. \(K(\sigma) \leq 0\) \(\forall \sigma\). Then \(M \cong \mathbb{R}^n\) — in fact \(\exp TM \to M\) is a diffeo.

**Proof.** We have already seen that \(F = \exp_m: TM \to M\) is a local diffeo. Induce on \(TM\) a Riem. metric by \(F^{-1}(g)\). Note that the slices through \(O_m\) are bimetric geodesics — geodesics defined \(\forall t\).

By Hopf-Rinow Thm, \(TM = (\mathbb{R}_+ Tm, F^t)\) is complete, and \(F: TM \to M\) is a local isometry (by continu.)

Given \(m \in M\), suppose \(B_g(m)\) is a normal ball about \(m\), observe that \(A x \in F^{-1}(m)\), we have — we used completeness to guarantee \(\exp_x\) is defined in \(B_{\varepsilon_x}(0_m)\)

\[
\begin{align*}
B_{\varepsilon_x}(0_x) & \xrightarrow{dF_x} B_{\varepsilon_x}(0_m) \\
\exp_x & \downarrow \quad \circ \quad \downarrow \exp_m \\
B_{\varepsilon_x}(x) & \xrightarrow{F} B_{\varepsilon_x}(m)
\end{align*}
\]

Since \(dF_x\) is \(1-1\), \(F\) is \(1-1\).

Claim: If \(x, x' \in F^{-1}(m)\), \(B_{\varepsilon_x}(x) \cap B_{\varepsilon_x}(x') = \emptyset\)

**Proof.** If not, \(\exists g\)-geod. from \(x\) to \(x'\) w/ length < \(\varepsilon\). \(\lambda: [0, 1] \to TM\) \(\lambda(0) = x, \lambda(1) = x'\)

\[
F \circ \lambda: [0, 1] \to M\text{ a geodesic w/ length } < \varepsilon \text{ w/}
\]

\[
F(0) = F(x) = m = F(x') = F(1)
\]

but this contradicts the fact that \(B_g(m)\) is a normal ball.
Claim 2. If \( y \in F'(B_{\delta_0}(m)) \), then \( \exists x \in F'(m) \) st. \( y \in B_{\delta_0}(x) \).

Consequently, \( F'(B_{\delta_0}(m)) = \bigcup_{x \in F'(m)} B_{\delta_0}(x) \) and \( F' \) is a covering map.

Proof of Claim 2: the geodesic from \( F(y) \) to \( m \), \( \gamma : [0,1] \to M \) lifts to \( \tilde{T} \mathcal{N} \), \( \tilde{\gamma} : [0,1] \to \tilde{T} M \) by \( \tilde{\gamma}(t) = \exp_y (\mathbf{exp}_y^{-1}(x/\delta(t))) \) (which makes sense b/c \( T M \) is complete). \[ \square \]

Corollary: the universal covering of \( (M,g) \) complete w/ \( \kappa(\sigma) \leq 0 \) is \( \mathbb{R}^n \).

\[ \text{Ex:} \quad (M,g) \text{ complete Riem with non-constant } \kappa \leq 0 \]

\[ \mathbb{R}^n \xrightarrow{\text{exp}} T_m M \xrightarrow{\text{exp}} M \]

\[ F \text{ is the universal covering.} \]

Homework exercise: (write explicit form of Jacobi field in each case)

\[ F^k(y) = d \rho^2 + f_k (\rho) \gamma_0 \]

where \( \rho(x) = d(x,0) \), \( g_0 = \frac{1}{\rho^2} (g_E - d \rho^2) \) and

\[ f_k (\rho) = \begin{cases} \rho^2 & \text{if } K_0 = 0 \\ \frac{\sinh^2(\rho \sqrt{K_0})}{-K_0} & \text{if } K_0 < 0. \end{cases} \]

Theorem: Suppose \( (M,g) \) has constant curvature \( K_0 \) and is complete.

- If \( \kappa = 0 \), then the universal cover \( \tilde{S}^2 \) is Euclidean space \( \mathbb{R}^n \).
- If \( K_0 = -1 \), then \( \tilde{S}^2 \) is hyperbolic space \( \mathbb{H}^n \).
- If \( K_0 > 0 \), then \( \tilde{S}^2 \) is a scaling of hyperbolic space \( \mathbb{H}^n \).
to see this, we observe, for \( K_0 = -1 \), we get:

\[
F^* (g_{H^n}) = G^* (g_\rho), \quad F \text{ is a local isom.}
\]

\( F \) is a diffeo., \( G \) the universal covering \( G \circ F : H^n \to M \) locally isom. universal covering. \( \square \)

So \( M \cong \frac{H^n}{\Gamma}, \quad \Gamma \subset \mathrm{Isom} (H^n). \) Similarly, for \( \mathbb{R}^n \) the covering transformations are isometries. 

[see first homework]

**Theorem** If \((M, g)\) is complete w/ constant curvature \(1\), then the universal cover \( S^n \) pt. Suppose \((M, g)\) has constant curv. \( K_0 > 0 \). Using homework again,

\[
F : B_{\infty, K_0} (0) \to M, \quad \text{local diffeo and}
\]

\[
\mathbb{R}^n \ni F^* (g_\rho) = \rho^2 + \frac{\sin^2 (\rho \sqrt{K_0})}{K_0} g_\rho.
\]

Observe that \( F^* (g_\rho) \bigg|_{S^2_{\rho} (0)} \) has diameter \( \to 0 \) as \( \rho \to \infty \).

\( F \) extend to a homeo. from one pt compactification of \( B_{\infty, K_0} (0) \)

\( \overline{B_{\infty, K_0} (0)} \), which is \( \mathbb{S}^n \). We have

\[
\overline{B_{K_0} (0)} \to \mathbb{S}^n, \quad \text{isomorphic to} \quad B_{\infty, K_0} (0), \quad B_{\infty, K_0} (0) \text{ is the completion of} \quad B_{K_0} (0), \quad \text{so} 
\]

\( F \) is an isometry. 

\[
M = \mathbb{S}^n / \Gamma, \quad \Gamma \subset \mathrm{Isom} (\mathbb{S}^n)
\]

\( M = \mathbb{S}^n / \Gamma \) w/ \( \mathbb{R}^n \), we're done since it's an isometry. \( \square \)
Theorem (Bonnec, Myers) (M,g) complete Ricci mfd. Suppose
\[ \mathrm{Ric}_g(v) \geq \frac{\lambda}{2} > 0 \]
\[ \forall m \in M, \forall t \in \mathbb{R}. \text{ Then } M \text{ is } \text{compact}. \]
\[ \text{Corollary} \ (M,g) \text{ complete, } \mathrm{Ric} \leq a \text{ bar. Then } \int_M \text{ vol } \leq a. \]
\[ \text{Proof:} \ \text{Let } \gamma : [0,1] \to M \text{ be the } \text{min. convex}. \ \phi(g) \text{ satisfies some curvature hyp. as } g, \text{ so } \tilde{M} \text{ is } \text{convex}. \ \text{Covering gp acts prop. transit.} \]
\[ \text{Since } \tilde{M} \text{ is compact}. \]
\[ \text{Ric}_g \text{ is any } \text{ non-convex}, \text{ so } \text{same thing is the case } K(0) \leq \frac{\lambda}{2}. \]
\[ \text{Proof of theorem:} \ \text{Let } \gamma : [0,1] \to M \text{ be a geodesic of length } > \pi \lambda \text{. We will show that } \gamma \text{ is not minimizing.} \]
\[ \text{Since any two points are connected by a minimizing geodesic, such a geodesic must have length } < \pi \lambda \text{, and will be done.} \]
\[ \text{This is similar to before, we sketch the argument.} \]
\[ \text{Let } \xi^0, \ldots, \xi^n \text{ be Killing o.m. frame fields over } \gamma \text{ with } \gamma'(t) = \xi^0(t), \xi^1(t), \ldots, \xi^n(t) \]
\[ \text{At } \gamma(t) \text{ set } \]
\[ \xi_j^i = \tilde{M}(\pi^j)(\xi_j^0(t)) \]
\[ \text{As before, the } 2^{nd} \text{ variation } \delta \text{ energy for } \delta \text{ by } \xi_j^i \]
\[ E_{\delta}''(0) = 2 \int_0^1 \delta \left( \frac{\delta^2}{2} \right) \left( \pi^2 - \xi_i^0(t) \right) \xi_j^i(t) \left( \pi^2 \right) \left( \xi_j^i(t) \right) \left( \pi^2 \right) \left( \xi_j^i(t) \right) \right) \left( \pi^2 \right) \left( \xi_j^i(t) \right) \right) \text{ dt} \]
\[ \sum_{j=1}^{n} E_{\delta}''(0) \leq 2 \int_0^1 \delta \left( \frac{\delta^2}{2} \right) \left( \pi^2 - \xi_i^0(t) \right) \xi_j^i(t) \left( \pi^2 \right) \left( \xi_j^i(t) \right) \left( \pi^2 \right) \left( \xi_j^i(t) \right) \right) \left( \pi^2 \right) \left( \xi_j^i(t) \right) \right) \text{ dt} \leq 0 \]
\[ E_{\delta}''(0) < 0 \text{ for some } j, \text{ as before, it follows } \delta \text{ is not locally minimizing } \delta, \text{ hence not minimizing.} \]
Under the assumption of positive sectional curvature, one has:

**Theorem (Synge, Weyl)*** If \((M^n, g)\) is complete Riemannian with positive sectional curvature \(K_{ij} \geq \frac{1}{n} > 0\), then

1. If \(n\) is even and \(M\) is orientable, then \(\pi_1 M = \mathbb{Z}\).
2. If \(n\) is odd, then \(M\) is non-orientable.

Proof: See DC 203-207

\[ \exists \Pi \subset \mathbb{R}^n = \mathbb{R}^{n-1}, \quad \{ \pm 1 \} < O_n = \text{Isom} (S^n). \]

In odd dimension:

\[ \mathbb{S}^{2n+1} = \{ x \in \mathbb{C}^{n+1} | |x| = 1 \} \]

Let \( p_0 \mapsto p_1 \in \mathbb{R}_+, \quad \text{gcd}(p_1, q) = 1 \) and consider:

\[ \mathbb{Z}_2 \times \mathbb{S}^{2n+1} \longrightarrow \mathbb{S}^{2n+1} \quad \Rightarrow \quad \mathbb{Z}_2 \longrightarrow U_{n+1} \subset SO_{2n+2} = \text{Isom}^+ (S^{2n+1}) \]

\[ \lambda (p_0, p_1) = \frac{2n+1}{p_1} \quad \text{Lens Space} \quad \pi_1 (L(p_0, p_1)) = \mathbb{Z}_2. \]

There are examples of space forms: A Riemannian mfd \(M\) with constant curvature is called a space form and as we have already noted, is a quotient of

\[ \widetilde{M} = \mathbb{H}^n, \mathbb{R}^n, \text{or } \mathbb{S}^n \text{ (after normalization curvature } -1, 0, 1) \] by a subgroup \( \Gamma \subset \text{Isom} \widetilde{M} \),

acting properly discontinuous and freely.

**Proposition:** Let \((M, g)\) be a Riemannian mfd, \( \Gamma \subset \text{Isom} (M, g) \) acts properly discontinuous on \(M\) if and only if \( \Gamma \) is discrete.

To prove this, we will need the following, which is of independent interest.

If \(G\) acts on \(M\) and \(N\), then a map \(f: M \to N\) is equivariant with those actions if \(g \in G, m \in M\), we have:

\[ g^* (f(m)) = f(g \cdot m). \]
Last time we discussed some examples for $\mathbb{R}^2$. 

A general construction: shortly, but first, some concrete examples.

Consider $\mathbb{H}^2$, several models: hyperboloid, upper half-plane, isometric. Since $k=-1$, also have unit disk $\mathbb{H}^2 = \{ z \in \mathbb{C} \mid |z|<1 \}$, $d_{\mathbb{H}^2} = \frac{\sqrt{1-|z|^2}}{|z|} d_E$. - Conformal.

Geodesics are (up to parametrization) arcs of lines and circles at $\mathbb{S}^1$.

A $1/2$-plane is a subset consisting of all points on one side of a geodesic.

A polygon is the intersection of planes where $\partial$'s are a locally finite set of geodesics.

A $(n+1)$-dimensional manifold is obtained by considering the $1$-parametric family of regular $n$-gons.

"Centered" at $x$ if $0 < t < 1$.

Note: $Q_2 = \text{Isom}(\mathbb{H}^2)$

Let $\Theta_n(t) = \text{interior of angle } t$ clearly varies continuously with $t$.

$t \to 0$, $\Theta_n(t) \to \text{interface of regular } n+1$-gon $\implies [\mathbb{R}^2 - \text{scale up}] \implies [\Theta_n(t) \to 0]$ as $t \to 1$. $\Theta_n(t) \to 0$.
By the intermediate value theorem, there is some \(0 < t < 1\) st.
\[
0 < \frac{\pi}{qg} = \frac{\pi}{qg} < \frac{q+2\pi}{qg}.
\]
Let \(\mathcal{P}_q\) be this polygon.

Label the sides \(\mathcal{P}_q\) of \(X_i\), then consider the unique isometries \(T_1, T_2\) where \(T_1(x_i) = x_{i+2}\) and \(T_2(x_i) = x_{i+2}\) for \(\mathcal{P}_q\). Such \(T_i\) exist b/c \(\text{Isom}(\mathbb{H}^2)\) acts transitively on \(\mathbb{H}^2\) w/ \(\text{stab}(z) = O_2 \trianglelefteq \text{Isom}(\mathbb{H}^2)\).

Consider the surface obtained by identifying opposite sides via \(T_1\); \(\mathcal{S} = \mathcal{P}_q / \sim\) where \(\sim T_i(x)\) \(\forall x \in X_i\) and all vertices are equal.

There is a metric on \(\mathcal{S}\) making it locally isometric to \(\mathbb{H}^2\):
\[x \in \mathcal{S}\] \(\sim T_i(x)\); \(\bigcirc\) \(x\)-vertex; \(\bigcirc\)

These local isometries \(\sim\) give \(\mathcal{S}\) to \(\mathbb{H}^2\); exhibit \(\mathcal{S}\) as a smooth surface.

(4) Prove that the metric is induced by a Riemann metric w/ constant curvature \(K = -1\).

\[
\mathcal{S} = \mathbb{H}^2 / \Gamma.
\]
\[
\Gamma = \langle T_1, T_2 \rangle < \text{Isom}(\mathbb{H}^2)
\]

Every closed surface \(\mathcal{S}\) of genus \(g \geq 2\) admits a hyperbolic structure — a metric \(\gamma\) curvature \(-1\).

**Theorem:** If \(\mathcal{S} = \mathbb{H}^2 / \Gamma\) is a hyperbolic surface of genus \(g \geq 2\), then area \(\mathcal{S} = \frac{8\pi}{(g-1)}\).

**Proof:** Let \(z \in \mathbb{H}^2\), consider
\[
D_\gamma(z) = \{ w \in \mathbb{H}^2 | d(z, w) \leq d(z, \infty) \forall \in \Gamma \}
\]
\[
= \{ w \in \mathbb{H}^2 | d(z, w) \leq d(z, \infty) \forall \in \Gamma \}
\]
Action prop. case \( \Rightarrow \) \( \Gamma \) is discrete,
\[ D_\gamma(z) = \bigcap_{w \in \Gamma} H(z, wz) \]
\[ H(z, wz) = \{ w = e^{i\theta} | d(z, w) < d(z, wz) \} = \mathbb{H} - \text{plane} \]

\( D_\gamma(z) \) is a vert poly gen and \( S = D_\gamma(z) / \Gamma \):

- Every orbit has - exp. in \( D_\gamma(z) \), only one trivial point has exactly one.
- \( d_S(\gamma(z), \gamma(w)) = d(z, w) \) \( \forall \) \( \gamma \in D_\gamma(z) \) since \( \gamma \) is a close pt to \( z \) in \( \Gamma \).

- \( D_\gamma(z) \subset \text{diam}(S) - \text{ball} \) about \( z \).

Now, subdivide \( D_\gamma(z) \) into triangles

**Exercise:** Area \( (\Delta(a, b, c)) = \pi - (a+b+c) \) where \( a, b, c \) are interior angles.

Let: assume \( a = 0 \) "vertex at \( 0 \)" in upper half plane.

Apply triple integral compute the integral \( \iint_{\Delta} \frac{dxdy}{x^2 + y^2} = \pi - (a+b+c) \)

In general:

\[ \pi - (a+b+c) = \pi - (a+c) + \pi - (a+c) \]

"Triangulation" of \( S \) with \( V = \# \text{vertex}, E = \# \text{edges}, F = \# \text{faces} \). \( 3F = 2E \)

\[ 2g - 2 = -V + E - F \]
\[ 4g - 4 = -2V + 2E - 2F \]
\[ = F - 2V \]

Area \( (S) = \sum_{i} \text{Area} (\Delta_i) \)
\[ = \sum_{i} \left( \pi - \left( q_i^1 + q_i^2 + q_i^3 \right) \right) = \chi - \sum_{i} \left( q_i^1 + q_i^2 + q_i^3 \right) 
= \chi - 2\pi V = \pi (F - 2V) = (g - 1)\pi \]
Dimension of the space of hyperbolic structures on a surface

We can get a rough estimate of this as follows:

Write $S = \mathbb{H}^2 / \Gamma$ and let $\rho: \pi_1(S) \to \mathbb{H}^2$ be the associated isomorphism.

Then $\rho \in \text{Hom}(\pi_1(S), \text{Isom}(\mathbb{H}^2)) \setminus \{1\}$ acts image acting properly and freely.

\[ \text{Hom}(\pi_1(S), \text{Isom}(\mathbb{H}^2)) = \{ (x_1, y_1, \ldots, x_g, y_g) \in (\text{Isom}(\mathbb{H}^2)^{2g} \setminus \{1\} \mid \prod_{i=1}^{g} [x_i, y_i] = 1 \} \]

When $\pi_1(S) = \langle a_1, b_1, \ldots, a_g, b_g \mid \prod_{i=1}^{g} [a_i, b_i] = 1 \rangle$

\[ \cong \text{nbhd of } \rho \text{ st. } \forall \rho \in U \cdot \rho(\pi_1(S)) \text{ acts properly & freely.} \]

Fund. domain for $\rho$ close to Fund. domain for $\rho_0$. . .

If $\rho$ and $\rho'$ are conjugate, the $\mathbb{H}^2 / \rho(\pi_1(S))$ isometric to $\mathbb{H}^2 / \rho(\pi_1(S))$

So, a nbhd of $\rho_0$ in $\text{Hom}(\pi_1(S), \text{Isom}(\mathbb{H}^2)) / \text{conj.}$ parametrizes deformations of hyperbolic metric on $S$

\[ \text{dim} = 3 \cdot 2g - (3 - 3) = 6g - 6 \]
We can also construct 3-mfD by similar procedure to gluing 3-disk hyperbolic polyhedron.

Ex: 1-parameter family \( D_t \) of hyperbolic regular dodecahedron:

- Dihedral angles \( \theta(t) \) vary continuously for \( 0 < t < 1 \):
  - \( \lim_{t \to 0} \theta(t) = 60^\circ \) angles \( \approx 110^\circ \)
  - \( \lim_{t \to 1} \theta(t) = 60^\circ \)

  For some \( t \), \( \theta(t) = 72^\circ \)

Identify opposite sides by heptading that twists \( \frac{3}{5} \) of the way around for all pairs of sides.

Each edge is identified with exactly 4 others, so get \( \frac{3}{5} \) of 5 edges gluing together.

Obviously etc for points or put \( D_t \) and on faces. Can check if 5

vertex as well — links have regular icosahedral

tessellation.