We wish to make precise the statement that the sectional curvature measures the rate at which geodesics diverge from each other.

How should we measure this? \((M,g)\) Riem ann. \(m \in M\)

\[\sigma \subset T_m M\] a 2-plane

\[\sigma \subset T_m M\]

\[\gamma(t)\]

\[\tau(t)\]

Circle in normal sphere,
Tangential vector to normal circle

Measure the rate of divergence using distance along circles in normal spheres.

Infinitesimally, this is described by the length of the tangential vector to this circle.

Recall Gauss lemma 2 (for proof)

Fix \(v, w\) on basis for \(\sigma \subset T_m M\). Let \(\theta(t)\) be a path in \(S^1(0)\)

\[S^1(0) \cap \sigma \subset T_m M \quad v / \theta(0) = 0 \quad \theta(0) = v\]

Parameterize the unit circle in \(v\)

Define \(f(t, u)\) by

\[f(t, u) = \exp_{v}(tu)\]

\[f(t, u)\]
We proved that \( \frac{\partial f}{\partial x}(x,0) \frac{\partial f}{\partial y}(x,0) \) is tangent to a radial geodesic.

Theorem: With the notation as above

\[
\left( \frac{\partial f}{\partial x}(t,0) \right)^2 = t^2 - \frac{1}{3} K(t) t^4 + \mathcal{E}(t)
\]

where \( \frac{\mathcal{E}(t)}{t^4} \to 0 \) as \( t \to 0 \).

Corollary: \( \frac{\partial f}{\partial x}(t,0) = t - \frac{1}{6} K(t) t^3 + \mathcal{E}(t) \) when \( \frac{\mathcal{E}(t)}{t^5} \to 0 \) as \( t \to 0 \).

In fact, parameterize the entire circle \( S^1(0,0) \) with unit speed the length of the circle in \( M \) is \( \frac{\partial f}{\partial x}(t,0) \) \( ds \). and if we knew \( \mathcal{E} = 3(2,1) \) \( \frac{\mathcal{E}(t)}{t^5} \to 0 \) as \( t \to 0 \) uniformly, \( \delta(t) \) then

\[
K(t) = \lim_{t \to 0} \frac{2}{\pi} \left( 2\pi t - \frac{\partial f}{\partial x}(t,0) \right) \quad \text{(see next homework sheet for details)}
\]

Sectional curvature is an infinitesimal measure of how much longer circles are in \( M \) than in \( \mathbb{R}^2 \).

To prove the theorem, we study the vector field along the geodesic \( y(t) \) defined as above — let \( v = \delta(0) \), \( w \in \mathbb{R} M \), \( \theta = p.X \), \( \theta(0) = w \), \( \delta(0) = v \).

Proposition: The vector field \( J(t) = \frac{\partial f}{\partial x}(t,0) \) satisfies

\[
\frac{D^2 J}{dt^2} + R(y,J)y' = 0.
\]

This is called the Jacobi equation.
Lemma \[ f: U \to M \text{ smooth, } \xi \in \mathfrak{X}(f) \text{ then} \]
\[
\frac{D}{dt} \frac{D}{dt} \xi - \frac{D}{dt} \frac{D}{dt} \xi = R(\xi, \xi) \xi .
\]

Proof: This is computation - see Theorem p98 (Ch. 4).

Idea: If \( \xi \in \mathfrak{X}(M) \) s.t. \( \frac{df}{ds} = \xi(f) \), \( \frac{df}{dt} = \xi(f) \), \( \xi = \xi(f) \)
then \( \frac{df}{dt} = \nabla_{\xi} f \) is related to \( \xi(f) \) respectively, so
\[ \frac{df}{dt} \xi(f) = 0. \]

Now this is basically the defn of curvature:
\[
\frac{D}{dt} \frac{D}{dt} \xi - \frac{D}{dt} \frac{D}{dt} \xi = \frac{D}{dt} \left( \left( \nabla_{\xi} \xi \right) \xi \right) - \frac{D}{dt} \left( \left( \nabla_{\xi} \xi \right) \xi \right) + \left( \nabla_{\xi} \xi \right) \xi \frac{df}{dt} .
\]

\[
= \left( \nabla_{\xi} \xi \right) \xi \frac{df}{dt} - \left( \nabla_{\xi} \xi \right) \xi \frac{df}{dt} = R(\xi, \xi) \xi \frac{df}{dt}
\]

\[ = R(\xi, \xi) \xi \quad \Box \]

Proof of Proposition: Since \( t \mapsto f(t, x) \) is a geodesic, \( \frac{D}{dt} \frac{df}{dt}(t, x) = 0 \)

\[
\Rightarrow 0 = \frac{D}{dt} \left( \frac{df}{dt} \right) = \frac{D}{dt} \frac{df}{dt} - R\left( \frac{df}{dt}, \frac{df}{dt} \right) \frac{df}{dt}
\]

\[
= \frac{D}{dt} \frac{D}{dt} \frac{df}{dt} + R\left( \frac{df}{dt}, \frac{df}{dt} \right) \frac{df}{dt}
\]

(old Lemma p. 77 of notes

Setting \( J(t) = \frac{df}{dt}(t, 0) \), \( Y(t) = f(t, 0) \) we get the desired equation

\[
\frac{d^2}{dt^2} J + R(Y, J) Y = 0. \quad \Box
\]

Any \( \xi \) satsifying the Jacobi eqn is called a Jacobi field.
We can pick a parallel on frame $\gamma$, e.g. $\gamma'(0) = e_1$.

If $J$ is a Jacobi field, $J = \sum f_j \xi_j$ then

$$\frac{d^2}{dt^2} J = \sum f_j'' \xi_j$$

and Jacobi eqn becomes

$$0 = \sum (f_j'' + g_j R_{iijj}) \xi_j$$

as 2nd order ode.

$$\Rightarrow \text{an linearly indep. sol.} \Rightarrow \text{soln. determined by}\$$

$$\text{init. values } \{ f_i(0), f_i'(0) \} \mapsto \text{soln. } \{ J(0), \frac{dJ}{dt}(0) \}$$

Note: $J(0) = \gamma(0)$ and $J'(0) = \gamma'(0)$ are solns., but not so interesting.

$$\begin{bmatrix} J = ce_1, J = te_1, \ldots \end{bmatrix}$$

$$\frac{d^2 J}{dt^2} = 0.$$  We'll consider Jacobi fields \underline{which are linear combinations of the above} more precisely, those $J \perp \gamma'$.

Also: Really only care about these coming from previous construction w/o $\gamma(u)$.

These are characterized by $J(0) = 0, \frac{dJ}{dt}(0) \perp \gamma$.

Proposition: If $J$ is a Jacobi field along geodesic $\gamma: [0, a] \to M$, $\gamma'(0) = 0$,

then

$$J'(t) = \frac{d}{dt} \theta(t)$$

where $\theta(t) = \exp_m (t \theta(0))$ when

$$\theta(0) = \gamma(0), \quad \theta'(0) = \frac{d\gamma}{dt}(0)$$

Equivalently,

$$J(t) = (d \exp_m) (t \theta(0))$$

where $d \exp_m$ is the differential of the exponential map at $m$.  \hfill $\blacktriangleleft$
proof: check some int. condition:

\[ J(0) = 0 = \frac{\partial f}{\partial x} (0, 0) \]

- def. \( f(0, x) = x f_0 \)

\[ \frac{\partial f}{\partial t} (0, 0) = (\text{det} \exp) \left( \frac{\partial f}{\partial t} (0) \right) = \frac{\partial f}{\partial t} (0) \]

\[ \left[ \frac{\partial^2 f}{\partial t^2} (1, 0) = \frac{\partial}{\partial t} \left( \text{det} \exp \right) \left( \frac{\partial f}{\partial t} (0) \right) = \frac{\partial}{\partial t} (\text{det} \exp) \left( \frac{\partial f}{\partial t} (0) \right) \right] \]

- def. \( f(0, x) = x f_0 \)

\[ \text{Constant curvature setting:} \]

Recall that if \( (M, g) \) has constant curvature \( K \), then

\[ g(R(\xi, \eta), \xi') = K <\xi, \xi'> <\eta, \eta'> - <\eta, \xi'> <\xi, \xi'> \]

Suppose \( \xi \) is a unit speed geodesic, \( J \perp \xi \) then \( \forall \xi \in \text{T}_\xi M \)

\[ g(R(\xi, J) \xi, \xi') = K <\xi, \xi'> <\xi, \xi'> - <\xi, \xi'> <\xi, \xi'> \]

\[ = K <\xi, \xi'> \]

(52) summing up over an orthonormal

\[ \Rightarrow R(\xi, J) \xi = K \xi \]

\[ \Rightarrow \text{Jacobi eqn:} \quad \frac{D^2}{\partial t^2} J + K J = 0. \]

If \( \xi(t) \) is any parallel field over \( \xi \) w/ \( |\xi(t)| = 1 \), then \( \text{Jacobi eqn:} \)

\[ J(t) = \frac{\text{sm}(t, K)}{\sqrt{K}} \xi(t) \]

- if \( K > 0 \)

\[ \frac{\text{sm}(t, K)}{\sqrt{K}} \xi(t) \]

- if \( K = 0 \)

\[ t \xi(t) \]

- if \( K < 0 \)

\[ \frac{\text{sm}(t, K)}{\sqrt{-K}} \xi(t) \]

\[ \Rightarrow \text{Jacobi eqn:} \quad f'' + K f = 0 \]
Theorem: Let $x: [0, \alpha] \to M$ be a geodesic, $m = x(0)$ and $J \in \mathfrak{X}(M)$ a Jacobi Field on $x$ w. $J(0) = 0$, $\frac{D^2 J}{dt^2}(0) = 1$. Then

$$|J(t)|^2 = t^2 - \frac{1}{2} g\left(R\left(x', x', J\right)x', \frac{D^2 J}{dt^2}\right)|J| + \varepsilon(t)$$

where $\varepsilon(t) \to 0$ as $t \to 0$.

Proof: $|J(t)|^2 = g(J(t), J(t))$. To find the coeff. of Taylor series, we evaluate at $t = 0$:

- $g(J(0), J(0)) = 0$
- $\frac{d}{dt} \left( g(J(t), J(t)) \right) = g\left( \frac{D}{dt} J, J \right) + g\left( J, \frac{D}{dt} J \right) = 2 g\left( \frac{D}{dt} J, J \right)$

$\Rightarrow \frac{d}{dt} \left( g(J(t), J(t)) \right) |_{t=0} = 0$

- $\frac{d^2}{dt^2} \left( g(J(t), J(t)) \right) = 2 g\left( \frac{D^2}{dt^2} J, J \right) + 2 g\left( \frac{D}{dt} J, \frac{D}{dt} J \right)$

$\Rightarrow \frac{d^2}{dt^2} \left( g(J(t), J(t)) \right) |_{t=0} = 0 + 2 = 2$

- $\frac{d^3}{dt^3} \left( g(J(t), J(t)) \right) = 2 g\left( \frac{D^3}{dt^3} J, J \right) + 6 g\left( \frac{D^2}{dt^2} J, \frac{D}{dt} J \right) + \sqrt{\text{Jacobi eqn}}

$\frac{d^3}{dt^3} \left( g(J(t), J(t)) \right) |_{t=0} = 0 + 0 + 0$

- $\frac{d^n}{dt^n} \left( g(J(t), J(t)) \right) = 8 g\left( \frac{D^n}{dt^n} J, J \right) + 8 g\left( \frac{D^{n-1}}{dt^{n-1}} J, \frac{D}{dt} J \right) + 6 g\left( \frac{D^{n-2}}{dt^{n-2}} J, \frac{D^2}{dt^2} J \right)$

$\frac{d^n}{dt^n} \left( g(J(t), J(t)) \right) |_{t=0} = 0 + 0 + 0$

For any $\delta \in \mathfrak{X}(M)$:

$$g\left( \frac{D^3}{dt^3} J, \delta \right) = g\left( \frac{D}{dt} \left( R\left( x', x', J \right)x', \delta \right), \delta \right) |_{t=0}$$

$$= \frac{d}{dt} g\left( R\left( x', x', J \right)x', \delta \right) |_{t=0} = g\left( R\left( x', x', J \right)x', \delta \right) |_{t=0} + g\left( R\left( x', x', J \right)x', \frac{D}{dt} \delta \right) |_{t=0}$$

$$= -g\left( R\left( x', J \right)x', x' \right) |_{t=0} + g\left( R\left( x', x', J \right)x', \frac{D}{dt} \delta \right) |_{t=0}$$

$$= -g\left( R\left( x', \frac{D}{dt} J \right)x', x' \right) + g\left( R\left( x', x', \frac{D}{dt} J \right)x', \delta \right) |_{t=0}$$
As a corollary, we obtain an origonal motivating term.

Corollary: \( x, t \in \mathbb{R}, u/18(t)^{-1} = 1, \quad \frac{d}{dt} x = x' \). Then

\[
|J(t)|^2 = t^2 = \frac{1}{2} K(t) t^2 + \varepsilon(t)
\]

where \( t = \operatorname{span} \langle e_0, \frac{d}{dt} x \rangle \).

\[\text{Def. If} \quad \gamma : [0,a] \to M \text{ is a geodesic, then } \gamma(a) \text{ is a conjugate point of } \gamma(0) \text{ (along } \gamma) \quad \text{if } \exists \text{ a Jacobi field } J \in X(\gamma) \text{ with }
\]

\[J(0) = 0, \quad J(a) = 0, \quad J \text{ not identically } 0.
\]

Because such a Jacobi field has the description as

\[J(t) = (d\exp x_0)(t \frac{d}{dt} x_0)
\]

It follows that \((d\exp x_0)(x_0)\) is singular iff \( \gamma(a) \) is a conjugate point of \( \gamma(0) \).

That is, \( \exp x \) is a local diffeomorphism on a nbhd of \( x \in TM \) iff \( \exp x \) is not a conjugate of \( M \).

As a consequence, we have

Proposition: Let \((M,g)\) be a Riemannian manifold of non-positive sectional curvature; \( K(x) \leq 0 \) \( \forall x \), such that \( \exp x \) is defined on all \( TM \). Then \( \exp x \) is a local diffeomorphism,

That is, \( \exp x \) is a local diffeomorphism on \( M \). Need to check \((d\exp x)(x_0)\) is nonsingular \( \forall x_0 \in TM \),

then we can apply the inverse function theorem. Equivalently, need to show that if \( \gamma : [0, a] \to M \) is a geodesic, \( J \in X(\gamma) \) a Jacobi field with \( J(0) = 0 \), \( J \) not identically \( 0 \).
then \( J(t) > 0 \), we show \( J(t)/t^2 \to \infty \) \( t \to 0 \). From
the corollary above, we know \( 1J^3(0) = 1J^3(\bar{x}) = 0 \) (for small
\( t > 0 \), \( J(t) > 0 \) suffices to choose \( 1J^2(\bar{x}) \geq 0 \). \( \bar{x} \to 0 \). We Have:
\[
1J^2 = 2g(\frac{\partial J}{\partial t}, \frac{\partial J}{\partial t}) - 2g(\phi(\bar{x}, J) \delta J, J) = 2|\frac{\partial J}{\partial t}|^2 - 2K(\delta J) |\delta J|^2 \geq 0.
\]

\[\text{Energy functional and geodesics: (Skip to part 9.4.9, DC)}\]

Next we would like to give another characterization of geodesics as
critical points for the energy functional:
The energy of a piecewise smooth path \( \gamma : [\alpha, \beta] \to M \) is defined to be:
\[
E(\gamma) = \int_0^1 |\dot{\gamma}(t)|^2 \, dt.
\]

The relationship with length is:
\[
\ell(\gamma) = \left( \int_0^1 |\dot{\gamma}(t)| \, dt \right)^2 \leq \int_0^1 |\dot{\gamma}(t)| \, dt \int_0^1 |\dot{\gamma}(t)|^2 \, dt = \alpha \cdot E(\gamma)
\]
with equality iff \( |\dot{\gamma}(t)| \) is constant.

If \( \gamma : [\alpha, \beta] \to M \) is a geodesic, then \( \ell(\gamma)^2 = \alpha E(\gamma) \). If \( \gamma \) is minimizing:

Then \( E(\gamma) = \frac{\ell(\gamma)^2}{\alpha} \leq \frac{\ell(c)^2}{\alpha} \leq E(c) \) for any other curve \( c : [\alpha, \beta] \to M \) connecting \( \gamma(\alpha) = c(\alpha) \), and equality holds if \( \gamma \) is a geodesic. [Parameterized]

We can think of \( E \) as a function on the space of all piecewise smooth
paths \( \Sigma \) to \( M \):
\[ E : \Sigma \to \mathbb{R} \]

(Minimum geodesics are minima for \( E : \Sigma_{mp} = \{ \gamma : [\alpha, \beta] \to M/\gamma(\alpha) = m, \gamma(\beta) = p \} \to \mathbb{R} \). \)
We would like to compute the derivative $S^1 \to \mathbb{S}^2 \to \mathbb{R}$ (or $E: S^2 \to \mathbb{R}$).

What is a tangent vector to $\mathbb{S}^2$ at $\varphi(\zeta)$?

Ans-a vector field $\xi \in \mathfrak{X}(\mathbb{S}^2)$.

A "path in $\mathbb{S}^2$ through $\varphi$" is a map $f: \mathbb{R} \to \mathbb{S}^2$ with $f(0, \zeta) = \varphi(\zeta)$.

This is called a variation of $\varphi$, and the associated vector field

$$\xi(\zeta) = \frac{\partial f}{\partial t}(0, \zeta)$$

(a "tangent vector" to $\mathbb{S}^2$ at $\varphi$) is called the variational field of $\xi$.

Any $\xi \in \mathfrak{X}(\mathbb{S}^2)$ defines some variation, by

$$f(\zeta, t) = \exp_{\varphi(\zeta)}(t \xi(\zeta))$$

(tout of course, it is not unique). We write $f_\xi(t) = f(\zeta, t)$.

The derivative $S^1 \to E$ in the direction $\xi$ is defined as $E_\xi(\varphi)$, where

$$E_\xi(\varphi) = E(f_\xi)$$

and $f$ is a variation of $\varphi$ with variational field $\xi$.

Theorem: If $f$ is a smooth variation of smooth $\varphi$, with variational field $\xi$, then

$$E_\xi(\varphi) = -\frac{1}{2} \left[ \int_0^1 g(\dot{\varphi}(\zeta), \frac{\partial \varphi}{\partial t}(\zeta)) \, dt + g(\varphi(1), \varphi(1)) - g(\varphi(0), \varphi(0)) \right]$$

(If only piecewise smooth, then there are other correction terms see $\partial D(\varphi)$.)

Proof: $E_\xi(\varphi) = \int_0^1 g(\frac{\partial f}{\partial t}, \frac{\partial f}{\partial t}) \, dt$ so

$$\frac{d}{dt} E_\xi(\varphi) = \int_0^1 \frac{\partial g}{\partial t}(\frac{\partial f}{\partial t}, \frac{\partial f}{\partial t}) \, dt = 2 \int_0^1 g(\frac{\partial^2 f}{\partial t^2}, \frac{\partial f}{\partial t}) \, dt = 2 \int_0^1 g(\frac{\partial f}{\partial t}, \frac{\partial f}{\partial t}) \, dt$$

$$= 2 \left[ \int_0^1 g(\frac{\partial f}{\partial t}, \frac{\partial f}{\partial t}) \, dt - \int_0^1 g(\frac{\partial f}{\partial t}, \frac{\partial f}{\partial t}) \, dt \right] = 2 \left( g(\frac{\partial f}{\partial t}, \frac{\partial f}{\partial t}) \right)_0^1 - \int_0^1 g(\frac{\partial f}{\partial t}, \frac{\partial f}{\partial t}) \, dt$$
So \( E_f(0) = -2 \left(\int_0^a \left( g(\frac{\partial f(t)}{\partial t}, \frac{\partial t}{\partial t} (0,0)), \frac{\partial f}{\partial x}(0,0) - g(\frac{\partial f(t)}{\partial t}, \frac{\partial f}{\partial y}(0,0)) \right) dt + g(\frac{\partial f(t)}{\partial t}, \frac{\partial f}{\partial x}(0,0)) \right) \)

Substituting \( \frac{\partial f}{\partial x}(0,0) = \xi(+) \) and \( \xi(+) = t(+) \) completes the proof [1].

A proper variation is a variation for which \( f(0,0) = f(0,0) \) and \( f(0,a) = f(0,a) \) for all \( a \). Then \( \xi(0) = 0, \xi'(a) = 0 \) such that the are tangent vectors to \( \Sigma_{mp} \). 2nd two terms on right \( \alpha = 0 \) and we have

**Corollary:** \( \psi [q, q] \rightarrow M \) is a geodesic iff for every proper variation field \( \xi \in \mathfrak{X}(\Sigma) \), \( E_\xi(0) = 0 \). That is, \( \psi \) is a normal parallel \( \xi \in \mathfrak{X}(\Sigma_{mp}) \rightarrow 1/\alpha \).

**Proof:** If \( \psi \) is a geodesic, then \( \frac{d}{dt} \psi(t) = 0 \Rightarrow \xi(t) = -2 \int_0^t \frac{d}{dt} \xi(t) dt = 0 \)

A proper variation fields \( \xi \).

Conversely, if \( E_\xi(0) = 0 \) for \( \xi \), then we \( \psi(t) = h(t), \frac{d}{dt} \psi(t) \) when \( h(t) > 0 \) for \( t > (\alpha, b) \), \( h(0) = h(\alpha) = 0 \), hence \( \psi(t) \) is a geodesic. So

\[
0 = E_\xi(0) = -2 \int_0^a h(t) \frac{d}{dt} \psi(t) dt = -2 \frac{d}{dt} \psi(t) |_{t=0} \Rightarrow \frac{d}{dt} \psi(0) = 0 \Rightarrow \]

**Remark:** only really need variation to be piecewise smooth.

**Remark:** If \( \psi \) is geodesic, then \( \mathfrak{J} \in \mathfrak{X}(\Sigma) \) is a Jacobi field iff \( \exists f(0,t) \) a variation \( \frac{\partial f(t)}{\partial t} = \mathfrak{J}(t) \) at \( \forall t, f(0,t) \) is a geodesic.

[see Homework]
Minimizing geodesics are minima for \( E: \Sigma_{m,p} \to \mathbb{R} \).

Call a geodesic \( \gamma: \mathbb{R} \to M \) connecting \( m \) to \( p \) locally minimizing if it minimizes length \( L(\gamma, \epsilon) \) for \( \epsilon \to 0 \) small, and all proper variations \( f \) of \( \gamma \). [Don't confuse \( L \) locally length minimizes any geodesic.]

Similar to above, it follows that \( \gamma \) is a local minimum for \( E \) on \( \Sigma_{m,p} \) iff \( \gamma \) is locally minimizing.

**Example:** \( S^n \), \( \gamma: \mathbb{R} \to S^n \) any geodesic with length \( > \pi \) all shorter than \( \gamma \).

Since geodesics \( \gamma: \mathbb{R} \to M \) are critical points of \( E: \Sigma_{m,p} \to \mathbb{R} \), we can try to decide if \( \gamma \) is locally minimizing by computing a 2nd derivative.

**Theorem** Let \( \gamma \) be a geodesic and \( f \) a proper variation with variational field \( \xi \), so \( E'(\gamma) = E'_f(\xi) = 0 \). Then

\[
E''_f(\gamma) = -2 \int_0^q g(\xi(\tau), \frac{\partial^2}{\partial t^2} \xi + R(\gamma', \xi)\gamma''(\tau) d\tau)
\]

when \( R \) is the curvature.
Proof. Just differentiate: \( \frac{d}{dt} \left( \frac{\partial}{\partial t} E'_t \right)(0) = \frac{d}{dt} \left( E'_t \right)(0) \)
\[
\frac{d}{dt} \left( E'_t \right) = \frac{2}{\alpha t^2} \left( \int_0^\alpha g \left( \frac{\partial f}{\partial x} + \frac{\partial f}{\partial t} \right) dt \right) \]
\[
= \frac{2}{\alpha t^2} \left( g \left( \frac{\partial f}{\partial x} + \frac{\partial f}{\partial t} \right) \right)_0^\alpha \]
\[
= \frac{2}{\alpha t^2} \left( g \left( \frac{\partial f}{\partial x} + \frac{\partial f}{\partial t} \right) \right)_0^\alpha \]
\[
= -\frac{2}{\alpha t^2} \left( \int_0^\alpha g \left( \frac{\partial f}{\partial x} + \frac{\partial f}{\partial t} \right) dt \right) + g \left( \frac{\partial f}{\partial x} + \frac{\partial f}{\partial t} \right)_0^\alpha \]

\( t = 0; \quad \alpha = 0 \)

proper variation

\[ \phi \in C^1 \quad \Rightarrow \quad \frac{D \phi(t)}{dt} \bigg|_{t=0} = 0 \]

\[
\frac{d^2}{dt^2} E(t) = \frac{d}{dt} \left( E'_t \right)(0) = -2 \int_0^\alpha g \left( \frac{\partial f}{\partial x} + \frac{\partial^2 f}{\partial x \partial t} \right) dt \quad (0)
\]
\[
= -2 \int_0^\alpha g \left( \frac{\partial^2 f}{\partial x \partial t} + R \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial t} \right) \right) dt \quad (0)
\]
\[
= -2 \left( \int_0^\alpha g \left( \frac{\partial^2 f}{\partial x \partial t} + R \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial t} \right) \right) dt \quad (0) \right)
\]

EX: Suppose \( \xi_0 \in \mathcal{Y}(0) \) is a parallel u.f. along \( \varphi \). \( \varphi : (0, 1) \rightarrow \mathcal{M} \) w/ \( \nabla \varphi \perp \xi_0(t) \) at \( t \), consider the proper variation:

\[ \xi'(t) = \sin(\pi t)\xi_0(t) \]

\( \therefore \quad \frac{d^2}{dt^2} \xi(t) = -\pi^2 \sin(\pi t) \xi_0(t) \]

Then:

\[ E''_\xi(t) = -2 \int_0^\alpha g \left( \xi(t), \frac{\partial^2 \xi}{\partial t \partial x} + R \left( \xi(t), \xi(t) \right) \right) dt \]
\[
= -2 \left( \int_0^\alpha g \left( \xi(t), \frac{\partial^2 \xi}{\partial t \partial x} + R \left( \xi(t), \xi(t) \right) \right) dt \right)
\]
\[
= 2 \pi^2 \int_0^\alpha g \left( \xi(t), \xi(t) \right) dt - 2 \frac{d}{dt} \int_0^\alpha g \left( \xi(t), \xi(t) \right) dt
\]
\[
= 2 \pi^2 \int_0^\alpha \xi(t) \xi(t) \sin(\pi t) dt - 2 \int_0^\alpha g \left( \xi(t), \xi(t) \right) dt = \int_0^\alpha \frac{\partial}{\partial t} R \left( \xi(t), \xi(t) \right) dt
\]
So, for manifolds with positive sectional curvature \( \langle g \rangle_{K_0} > 0 \), sufficiently long geodesics (at least \( \frac{\pi}{\sqrt{K_0}} \)) are not locally minimizing. \( \Lambda \) [We'll see that Ricci curvature bounds, in fact, suffices]

Sharp for \( \Lambda \).

In order to make this very useful, need to restrict to manifolds for which any two points can be connected by a geodesic.

**Lecture 19**

Hopf--Rinow & Hadamard Theorem Ch 7 DC

To use curvature hypotheses to study a Riemannian mfd, we must impose some further assumptions. If \((M, g)\) is Riemannian mfd, any curvature assumption on \(M\) is automatically valid for \(M - A\), where \(A \subset M\) any closed subset.

For example, assuming \(M\) is compact, connected, eliminates this type of phenomena.

This is too restrictive for many applications — e.g., the universal covering of a compact manifold need not be compact.

A weaker hypothesis, completeness, enjoys many of the same properties, and is applicable in more situations (as well see).

Assume from now on (unless o.w. stated), mfd\'s are connected.