1. Do the following problems from DoCarmo: 2, 6, 7 from Chapter 5, 2, 7, 12 from Chapter 7, and 5 from Chapter 9.

2. Consider $\mathbb{R}^n$ with the Euclidean metric $g_E$ and let $\rho : \mathbb{R}^n \to \mathbb{R}$ be the distance to the origin, $\rho(x) = |x|$. Away from 0, $\rho$ is smooth, and $d\rho^2 = d\rho \otimes d\rho$ is a symmetric 2–tensor. Furthermore, $g_E - d\rho^2$ is also a symmetric 2–tensor, and we write

$$g_S = \frac{1}{\rho^2}(g_E - d\rho^2).$$

(Observe that $g_S$ is not a Riemannian metric because it is not positive definite.) If $S_r$ is the sphere of radius $r > 0$ centered at 0, prove that

$$g_S|_{S_r} = \frac{1}{r^2}g|_{S_r}.$$

Prove that $d\rho^2 + g_S$ is a metric on $\mathbb{R}^n - \{0\}$ and that

$$f : \mathbb{R}^n - \{0\} \to (0, \infty) \times S^{n-1}$$

given by $f(x) = (|x|, x/|x|)$ is an isometry (where the target is given the product metric with the obvious metrics on the factors).

3. With notation as in the previous problem, note that on $\mathbb{R}^n - \{0\}$, $g_E = d\rho^2 + \rho^2 g_S$.

Now let $(M, g)$ be a Riemannian manifold of constant curvature $K$, $m \in M$ and $F : B_{\epsilon}(0) \to B_{\epsilon}(m)$ the inverse of normal coordinates for some $\epsilon$, where $B_{\epsilon}(0) \subset \mathbb{R}^n$ and $B_{\epsilon}(m) \subset M$. So, $F$ is a diffeomorphism sending radial lines from 0 isometrically to radial geodesics from $m$. Prove that

$$F^*(g) = d\rho^2 + f_K(\rho)g_S$$

where

$$f_K(\rho) = \begin{cases} \sin^2(\rho\sqrt{K}) \quad & \text{if } K > 0 \\ \rho^2 \quad & \text{if } K = 0 \\ \sinh^2(\rho\sqrt{-K}) \quad & \text{if } K < 0 \end{cases}$$

Hint: Use the construction of Jacobi fields for constant curvature manifolds given in Lecture 16 and their description as $(d\exp_m)_*(tv)$ for appropriate $v, w \in T_mM$.

As a consequence of this, prove that if $M$ and $N$ both have constant curvature $K$, then for any $m \in M, p \in N$, there exists $\epsilon > 0$ and an isometry $f : B_{\epsilon}(m) \to B_{\epsilon}(p)$. That is, any two manifolds of constant curvature $K$ are locally isometric.