1. Do problems 1, 6, 9 from Chapter 4 of Do Carmo.

2. Prove that if $M$ is a closed surface embedded in $\mathbb{R}^3$ with the induced Riemannian metric, then there is a point $m \in M$ where the Gauss curvature satisfies $K(m) \geq 0$.

3. Let $M \subset \mathbb{R}^3$ be a smooth orientable surface, and suppose $M$ is given the induced Riemannian metric. There is a smooth map $N : M \rightarrow \mathbb{R}^3$ which associates to each $m \in M$, a unit vector orthogonal to $T_m M \subset \mathbb{R}^3$.

(a) Prove that the derivative of this map

$$dN_m : T_m M \rightarrow \mathbb{R}^3$$

has $dN_m(T_m M) \subset T_m M$.

The negative of this derivative is called the *Shape operator* and we denote it

$$S_m = -dN_m : T_m M \rightarrow T_m M.$$  

(The sign choice is unimportant, but classical.) (b) Prove that the shape operator is symmetric:

$$g(S_m(v), w) = g(v, S_m(w)).$$

(c) Prove that the determinant of $S_m$ is precisely the Gauss curvature of $M$ at $m$.

**Remark [geometric interpretation of the shape operator].** Slice through $M$ with a plane perpendicular to $M$ at $m$. Near $m$ the intersection is a curve which we can parameterize $\gamma : (-\delta, \delta) \rightarrow M$ with $\gamma(0) = m$. We can assume this has unit speed so that $|\gamma'(t)| = 1$ for all $t$. Note that by the chain rule, $\gamma''(0) = \pm S_m(\gamma'(0))$. Or equivalently, the planar curve $\gamma$ has geodesic curvature $\kappa_\gamma(0) = |S_m(\gamma'(0))|$. Because $S_m$ is symmetric, it is diagonalizable. The eigenvalues are called the *principal curvatures* and they are the maximum and minimum value of the “signed” geodesic curvatures. The product of the principal curvatures is the Gauss curvature. So, if the principal curvatures have the same sign, then the Gauss curvature is positive. If they have opposite signs, the Gauss curvature is negative.

4. Let $(M, g)$ be a Riemannian manifold. For any constant $c > 0$, compute the curvatures of $(M, c^2 g)$ in terms of those for $(M, g)$ and $c$. Specifically, relate the tensors $R(\cdot, \cdot, \cdot)$ and $R(\cdot, \cdot, \cdot, \cdot)$, sectional curvature $K$, Ricci curvature $Ric$ and scalar curvature $Scal$ for $(M, g)$ with that for $(M, c^2 g)$. What is the sectional curvature of the sphere of radius $r > 0$?

5. Recall the Riemannian metric on an open subset $U \subset \mathbb{R}^n$ conformally equivalent to the euclidean metric—these have the form

$$g_{ij} = \frac{1}{F^2} \delta_{ij}$$

where $F$ is a positive smooth function on $U$. Setting $f = \log(F)$, we computed the Christoffel symbols (see lecture 8, page 38) of the notes.

Compute the curvatures. Verify that

$$R_{ij} = \frac{1}{F^2} R'_{ij} = \frac{1}{F^2} \sum_i f_{i}^2 + f_{i}^2 + f_{i}^2 + f_{i}^2 + f_{i}^2 + f_{ij}^2$$

for all $i, j$

$$R_{ijkl} = 0 \text{ if all indices are distinct}$$

while for any three distinct indices $i, j, k$ we have

$$R'_{ij} = -f_k f_j - f_{kj}, \quad R'_{ij} = f_i f_k + f_{ki}, \quad R'_{ij} = 0.$$ 

The sectional curvatures $K_{ij}$ in the direction $ij$ are given by

$$K_{ij} = F^2 \left( -\sum_i f_i^2 + f_i^2 + f_i^2 + f_i^2 + f_i^2 \right)$$

Using this and Corollary 3.5 of this chapter, prove that for our second model of hyperbolic space (which we haven’t yet verified is isometric to $\mathbb{H}^n$), which is the upper half space metric defined as above with $F = x_n$, we have constant curvature $-1$. (see Do Carmo page 160-162 to see this all worked out).