Math 519, Homework 2

1. Given two vector fields $\xi, \eta \in \mathfrak{X}(\mathbb{R}^n)$, viewing $\xi$ as a map $\xi : \mathbb{R}^n \to \mathbb{R}^n$ via the canonical identification $T_m \mathbb{R}^n = \mathbb{R}^n$, prove that

$$\nabla_\eta \xi (m) = (d\xi)_m (\eta_m)$$

satisfies

(a) $\nabla f \xi + \eta \xi = f \nabla \xi + \nabla \eta \xi$,
(b) $\nabla \eta (f \xi + \xi) = \eta (f) \xi + f \nabla \eta \xi$,
(c) $\nabla \eta \xi - \nabla \xi \eta = [\eta, \xi]$,
(d) $\xi (g(\eta, \xi)) = g(\nabla \xi \eta, \xi) + g(\eta, \nabla \xi \xi)$

That is, prove that $\nabla$ as defined above is the Levi-Civita connection on $\mathbb{R}^n$.

2. Do problems 1–3 and 8 from Chapter 2 of Do Carmo.

3. Let $\mathbb{R}^{n,1} = (\mathbb{R}^{n+1}, B_{n,1})$ denote Lorentz space, $\mathbb{H}^n \subset \mathbb{R}^{n,1}$ hyperbolic $n$–space and $\nabla$ denote the Levi-Civita connection on $\mathbb{H}^n$. Write $\Pi_m : \mathbb{R}^{n,1} \to T_m \mathbb{H}^n$ for the $B_{n,1}$-orthogonal projection

$$\Pi_m (v) = v + B_{n,1}(v, m) m$$

where we view $m$ as both a point in $\mathbb{H}^n$ and as a vector in $\mathbb{R}^{n,1}$. Prove that for any two vector fields $\xi, \eta \in \mathfrak{X}(\mathbb{H}^n)$ we have

$$\nabla_\xi \eta (m) = \Pi_m ((d\eta)_m (\xi_m))$$

where on the right hand side, we view $\eta$ as a map $\eta : \mathbb{H}^n \to \mathbb{R}^{n,1}$.

4. Do problems 2, 3, 5 from Chapter 3 of Do Carmo.

5. Let $S^n$ denote the unit $n$–sphere, and let $m \in S^n$ and $v \in T_m^1 S^n$. Prove that

$$\gamma(t) = \cos(t)m + \sin(t)v$$

is a unit speed geodesic through $m$ tangent to $v$. Consequently, the exponential map is given by

$$\exp(v) = \cos(|v|)m + \sin(|v|)v/|v|$$

for every $v \in T_m S^n$, $v \neq 0$.

6. Let $\mathbb{H}^n \subset \mathbb{R}^{n,1}$ denote hyperbolic $n$–space and let $m \in \mathbb{H}^n$ and $v \in T_m^1 \mathbb{H}^n$. Prove that

$$\gamma(t) = \cosh(t)m + \sinh(t)v$$

is a unit speed geodesic through $m$ tangent to $v$. Consequently, the exponential map is given by

$$\exp(v) = \cosh(|v|)m + \sinh(|v|)v/|v|$$

for every $v \in T_m \mathbb{H}^n$, $v \neq 0$. 