MATH 428

Final exam

due May 15, midnight

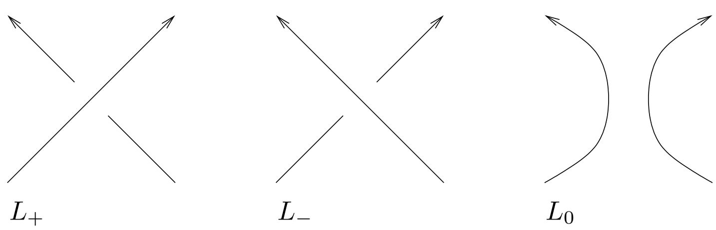
Instructions: Solve each problem as completely as possible, some are more difficult than others. You can rest assured that this is for my purposes only—how you perform on this exam will not have any effect on your grade. For this reason, I also ask that you limit your sources to the notes, Hatcher's book, and perhaps an elementary topology book (for example, Munkres, "Topology: A first course").

Have fun and have a good break!!

1. The HOMFLY polynomial of an oriented link L is a two variable Laurant Polynomial $P_L(\ell, m)$ defined by the two properties:

- $P_{\bigcirc}(\ell, m) = 1$, for the unknot $\bigcirc$, and
- If L_+, L_-, L_0 are three links that are identical outside a ball, and differ inside the ball as shown below, then

$$\ell P_{L_+}(\ell, m) + \ell^{-1} P_{L_-}(\ell, m) + m P_{L_0}(\ell, m) = 0$$



a. Compute $P_K(\ell, m)$ for K being the trefoil and the figure eight knot.

The following gives an alternative description of the Alexander polynomial:

Theorem. Substituting $\ell = i$ and $m = i(t^{1/2} - t^{-1/2})$ into P_L gives the Alexander polynomial of an oriented link L:

$$\Delta_L(t) = P_L(i, i(t^{1/2} - t^{-1/2})).$$

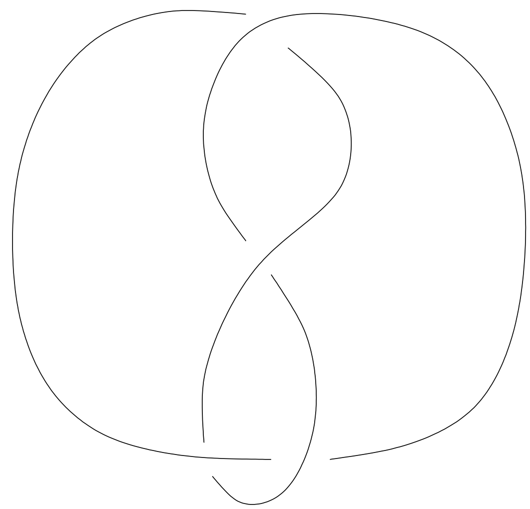
b. Using a previous computation of the Alexander polynomials, verify this theorem for the figure eight and the trefoil.

c. Prove that there is a substitution for ℓ and m which results in the Jones polynomial $V_L(t)$.

2. Let K denote the figure eight knot, and let X_K denote the exterior of K in S^3, $X_K = S^3 - N(K)$.

a. Compute a Wirtinger presentation for the fundamental group $\Gamma = \pi_1(X_K)$.

b. Solve for generators in terms of the others to find a presentation for Γ with exactly two generators.



c. Let $SL_2(\mathbb{C})$ be the group of 2×2 matrices with complex entries and determinant 1. Construct a homomorphism

$$\phi : \Gamma \to SL_2(\mathbb{C}).$$

so that the generators are sent to matrices of the form

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ -\omega & 1 \end{pmatrix}.$$

Moreover, using the fact that the relation in the presentation must be satisfied in $SL_2(\mathbb{C})$, prove that

$$\omega^2 + \omega + 1 = 0.$$

That is, ϕ sends Γ into the subgroup $SL_2(\mathbb{Z}[\omega])$, where $\mathbb{Z}[\omega]$ is the subring of $\mathbb{C}$ generated by $\mathbb{Z}$ and ω.

3. Let K denote a knot and $X_K = S^3 - N(K)$ the exterior of K, and $S \subset X_K$ a Seifert surface for K (thought of as a compact surface with boundary in X_K). Recall that S is a compact, connected orientable surface with exactly one boundary component. The following is a very deep result.

Theorem. If S has minimal genus among all Seifert surfaces and we let $i : S \to X_K$ denote the inclusion map, then $i_* : \pi_1(S) \to \pi_1(X_K)$ is *injective.*

Assuming this theorem, prove $\pi_1(X_K)$ is abelian if and only if K is the unknot.

4. a. Suppose that $S = S_g$, a compact, connected orientable surface of genus g and $p : \widetilde{S} \to S$ is an n–sheeted connected covering space for some $n < \infty$. Prove that $\widetilde{S}$ is orientable, and find its genus.

b. Prove that every compact connected non-orientable surface M_n has a 2-fold covering space $p : S \to M_n$ so that S is a compact connected orientable surface. What is the genus of S?

c. (harder) Suppose that $S = S_{g,1}$, a compact, connected orientable surface with one boundary component and $p : \widetilde{S} \to S$ is a finite sheeted regular covering. Prove that $\widetilde{S}$ has more than one boundary component.