1. Let $\mathbb{Q}$ denote the additive group of rational numbers and $H, K$ groups such that $\mathbb{Q} \cong H \times K$. Prove that either $|H| = 1$ or $|K| = 1$. That is, $\mathbb{Q}$ cannot be expressed as a nontrivial direct product. Can it be a nontrivial semi-direct product? Explain.

2. Let $G$ be a nonabelian finite simple group and let $p$ be the smallest prime dividing $|G|$. Prove that there is no subgroup of index $p$ in $G$.

3. Let $n$ be a positive integer and $p$ an odd prime with $p \leq n$.
   
   (i) Prove that every element of order $p$ in $S_n$ is an even permutation
   
   (ii) Prove that $A_n$ can be generated by $p$-cycles
   
   (iii) Show that the number of $p$-cycles in $S_n$ is given by
   
   $$L(n,p) = \binom{n}{p} (p-1)!$$
   
   (iv) Show that the number of elements of order $p$ in $S_n$ is
   
   $$\sum_{j=1}^{J} L(n,p)L(n-p,p)L(n-2p,p)\cdots L(n-(j-1)p,p) \frac{1}{j!}$$
   
   where $J$ is the greatest integer less than or equal to $n/p$.

4. Let $p$ and $q$ be distinct primes and $G$ a group of order $p^3q$.
   
   (i) Prove that $G$ either has a normal Sylow $p$ subgroup or a normal Sylow $q$ subgroup, unless $p = 2$ and $q = 3$. 

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(ii) Prove that $S_4$ does not have a normal Sylow 2 subgroup nor a normal Sylow 3 subgroup.

5. Let $G_1$ and $G_2$ be groups and suppose $G_1$ acts faithfully on a set $X_1$ and $G_2$ acts faithfully on a set $X_2$. If $|G_1| = |G_2| = 81$ and $|X_1| = |X_2| = 9$, prove that there exists an isomorphism $\phi : G_1 \to G_2$ and a bijection $f : X_1 \to X_2$ so that

$$f(g \cdot x) = \phi(g) \cdot f(x)$$

for all $x \in X_1$ and $g \in G_1$.

6. Let $G$ be a finite group, $H \leq G$ and $g \in G$. Consider the usual action of $G$ on the set of left cosets $G//H = \{xH\}_{x \in G}$, and let $n$ be the number of left cosets which are fixed by $g$:

$$n = |\{xH \mid gxH = xH\}|.$$

Prove

$$n = \frac{|C_G(g)| |[g] \cap H|}{|H|},$$

where $[g]$ denotes the conjugacy class of $g$ in $G$, and hence $|[g] \cap H|$ is the number of conjugates of $g$ that lie in $H$.

Hint: consider both the action of $G$ on $G//H$ as well as the action of $G$ on $[g]$ by conjugation.

7. Problem 18, section 4.4, page 138 (you may assume exercise 33 of section 4.3).