Local continuity implies continuity

We have

$$f : X \to Y$$

and we assume that

$$X = \bigcup_{\alpha \in J} U_\alpha$$

where $U_\alpha \subset X$ is open for all $X$, and $f|_{U_\alpha}$ is continuous for all $\alpha$. This means that for all $V \subset Y$ open, $f|_{U_\alpha}^{-1}(V)$ is open in $U_\alpha$ *in the subspace topology*. Now observe that since $U_\alpha$ is open in $X$, any open subset $W \subset U_\alpha$ is the intersection of an open set $W' \subset X$ with $U_\alpha$:

$$W = U_\alpha \cap W'.$$

Since the intersection of two open sets is open, $W$ is an open subset *of* $X$.

Now we observe that for any $V \subset Y$ we have

$$f^{-1}(V) = \bigcup_{\alpha \in J} (f^{-1}(V) \cap U_\alpha) = \bigcup_{\alpha \in J} f|_{U_\alpha}^{-1}(V)$$

So if $V$ is open, $f^{-1}(V)$ is a union of open sets, hence open, and $f$ is continuous.

The argument for $X = A_1 \cup \ldots \cup A_k$ is similar. We need to observe that a closed subset of a closed set in the subspace topology is closed in the whole space.