Differential Geometry: Levi-Civita connection

November 28, 2006

Theorem 6.5 *If g is a Riemannian manifold, then there exists a unique affine connection ∇ that is symmetric and compatible with g.*

Proof. If ∇ exists, then

$$\begin{aligned}\xi(g(\eta,\zeta)) &= g(\nabla_\xi\eta,\zeta)+g(\eta,\nabla_\xi\zeta)\\ \eta(g(\zeta,\xi)) &= g(\nabla_\eta\zeta,\xi)+g(\zeta,\nabla_\eta\xi)\\ \zeta(g(\xi,\eta)) &= g(\nabla_\zeta\xi,\eta)+g(\xi,\nabla_\zeta\eta)\end{aligned}$$

Adding the first two lines and subtracting the third gives

$$\xi(g(\eta,\zeta))+\eta(g(\zeta,\xi))-\zeta(g(\xi,\eta))$$
$$= g(\nabla_\xi\eta,\zeta)+g(\eta,\nabla_\xi\zeta)+g(\nabla_\eta\zeta,\xi)+g(\zeta,\nabla_\eta\xi)-g(\nabla_\zeta\xi,\eta)-g(\xi,\nabla_\zeta\eta)$$

Manipulating the right-hand side and applying symmetry we obtain

$$\begin{aligned}&= g(\nabla_\xi\eta,\zeta)+g(\zeta,\nabla_\eta\xi)+(\ g(\nabla_\xi\eta,\zeta)-g(\nabla_\xi\eta,\zeta)\ )\\ &\qquad +g(\eta,\nabla_\xi\zeta)-g(\nabla_\zeta\xi,\eta)+g(\nabla_\eta\zeta,\xi)-g(\xi,\nabla_\zeta\eta)\\ &= 2g(\nabla_\xi\eta,\zeta)+g(\nabla_\eta\xi-\nabla_\xi\eta,\zeta)+g(\nabla_\xi\zeta-\nabla_\zeta\xi,\eta)+g(\nabla_\eta\zeta-\nabla_\zeta\eta,\xi)\\ &= 2g(\nabla_\xi\eta,\zeta)+g([\eta,\xi],\zeta)+g([\xi,\zeta],\eta)+g([\eta,\zeta],\xi)\end{aligned}$$

Now we can solve for the only term involving ∇

$$\begin{aligned}g(\nabla_\xi\eta,\zeta) = \frac{1}{2}(\ &\xi(g(\eta,\zeta))+\eta(g(\zeta,\xi))-\zeta(g(\xi,\eta))\ )\\ &-g([\eta,\xi],\zeta)-g([\xi,\zeta],\eta)-g([\eta,\zeta],\xi)\end{aligned}$$

This uniquely determines ∇, so ∇ is unique.

On the other hand this last equation serves to **define** ∇. □

If we look in local coordinates $x^1,...,x^n$, and let $\xi=\frac{\partial}{\partial x^i}$, $\eta=\frac{\partial}{\partial x^j}$ and $\zeta=\frac{\partial}{\partial x^k}$, then the formula at the end of the proof becomes

$$g\left(\sum_\ell \Gamma^\ell_{ij}\frac{\partial}{\partial x^\ell},\frac{\partial}{\partial x^k}\right)=\frac{1}{2}\left(\frac{\partial}{\partial x^i}g_{jk}+\frac{\partial}{\partial x^j}g_{ki}-\frac{\partial}{\partial x^k}g_{ij}\right)$$

$$\sum_\ell \Gamma^\ell_{ij}g_{\ell k}=\frac{1}{2}\left(\frac{\partial}{\partial x^i}g_{jk}+\frac{\partial}{\partial x^j}g_{ki}-\frac{\partial}{\partial x^k}g_{ij}\right)$$

If we view (g_{ij}) as a matrix, then because inner products are positive (in particular, nondegenerate), there is an inverse (g^{ij}), i.e. $\sum_k g_{\ell k}g^{km}=\delta^m_\ell$. So

$$\Gamma^m_{ij}=\frac{1}{2}\sum_k\left(\frac{\partial}{\partial x^i}g_{jk}+\frac{\partial}{\partial x^j}g_{ki}-\frac{\partial}{\partial x^k}g_{ij}\right)g^{km}$$