Using Taylor series we can also get good estimates

for e:

$$e = e^1 = \sum_{n=0}^{\infty} \frac{1^n}{n!} = \sum_{n=0}^{\infty} \frac{1}{n!} \approx \sum_{n=0}^{N} \frac{1}{n!}$$

Not an alt. series (can't use integral test either..)

Can use Taylor's Theorem: $|f^{(n+1)}(x)| = e^x \le e < 3$ for $x \in [-1,1]$.

so error is bounded by

$$|R_N(1)| \le \frac{3}{(N+1)!} 1^{N+1} = \frac{3}{(N+1)!}$$

to get e within $\frac{1}{100,000,000}$, then we need

$$\frac{3}{(N+1)!} \le \frac{1}{100,000,000}$$

or

$$(N+1)! \ge 300,000,000$$

this happens when $N \ge 11$

so,

$$e \approx \sum_{n=0}^{11} \frac{1}{n!} = 2 + \frac{19,958,400 + 6,652,800 + 1,663,200 + 332,640 + 55,440 + 7,920 + 990 + 110 + 11 + 1}{39,916,800}$$

$$= 2 + \frac{28671512}{39916800} \approx 2.71828182|6198493$$

correct to here

We can use Taylor series to compute difficult/impossible integrals

Ex $\int e^{-x^2}\,dx$ has no expression in terms of elementary functions. However, observe that

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}$$ for every x, so

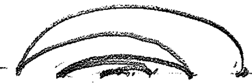
$$\int e^{-x^2}\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)\,n!} + C$$

We can also approximate definite integral using the series.

$$\int_0^1 e^{-x^2}\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)\,n!}\Bigg|_0^1 = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)\,n!} \approx \sum_{n=0}^{N} \frac{(-1)^n}{(2n+1)\,n!}$$

This is an alternating series that passes the alt. series test

Recall:



$*$ error in using the N^{th} partial sum is no more than the absolute value of $(N+1)^{st}$ term of series $*$

Error using $\sum_{n=0}^{N} \frac{(-1)^n}{(2n+1)\,n!}$ is at most $\frac{1}{(2(N+1)+1)(N+1)!} = \frac{1}{(2N+3)(N+1)!}$

so, to estimate within $\frac{1}{10{,}000}$, should take N so that

$$\frac{1}{(2N+3)(N+1)!} < \frac{1}{10{,}000} \quad \text{or} \quad (N+1)!\,(2N+3) > 10{,}000$$ — compute $N=6$ suffices.

Estimating functions by their Taylor polynomials is useful in many applications where exact values may not be required (though the precision may need to be specified), and polynomials are often simpler to work with.

EX $\sin(x) \approx x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$ how good is this on $[-1,1]$?

$\Rightarrow$ alt. series for any x and passes alt series test (terms decrease to 0)

so error is at most $\left|\frac{x^9}{9!}\right| \leq \frac{1}{362,880}$ for every $x \in [-1,1]$.
so within 5 dec. places.

Note adding one more term:

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!}$$

increases the precision to $\frac{1}{39,916,800}$ (or within 7 dec. places.

EX [student] $\sqrt{x}$ Taylor series at $a=4$? How well does degree 3 Taylor poly approx $\sqrt{x}$ on $[3,5]$?

$f(x) = \sqrt{x} = x^{1/2}$

$f'(x) = \frac{1}{2}x^{-1/2}$

$f''(x) = \frac{1}{2}\left(-\frac{1}{2}\right)x^{-3/2}$

$f'''(x) = \frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)x^{-5/2}$

$f^{(n)}(x) = (-1)^{n+1}\frac{1}{2^n}(1\cdot 3\cdot 5\cdots(2n-3))\,x^{-(2n-1)/2}$

$f^{(n)}(4) = -\left(-\frac{1}{2}\right)^n \frac{1}{(4)^{(2n-1)/2}} 1\cdot 3\cdot 5\cdots(2n-3)$

$= -2\left(-\frac{1}{2}\right)^n \frac{1}{4^n} 1\cdot 3\cdot 5\cdots(2n-3)$

$= -2\left(-\frac{1}{8}\right)^n 1\cdot 3\cdot 5\cdots(2n-3)$

$n \geq 1$

$f(4) = 2$

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(4)}{n!}(x-4)^n = 2 - 2\sum_{n=1}^{\infty}\left(\frac{-1}{8}\right)^n \frac{1\cdot 3\cdot 5\cdots(2n-3)}{n!}(x-4)^n$$

Radius of conv.?

Ratio test:

$$\lim_{n\to\infty} \frac{(1/8)^{n+1}}{(1/8)^n}\cdot\frac{1\cdot 3\cdot 5\cdots 2n-1}{1\cdot 3\cdot 5\cdots 2n-3}\,\frac{n!}{(n+1)!}\,|x-4| = \lim_{n\to\infty}\frac{|x-4|}{8}\,\frac{2n-1}{n+1} = \frac{|x-4|}{4} < 1$$

so $R = 4$

for $x \in [3,5]$, $$|f^{(n+1)}(x)| = \frac{1\cdot 3\cdot 5\cdots 2n-1}{2^{n+1}}\,\frac{1}{|x|^{\frac{2n+1}{2}}}$$

$$\leq \frac{1\cdot 3\cdot 5\cdots 2n-1}{2^{n+1}}\cdot\frac{1}{3^n} = \frac{1\cdot 3\cdot 5\cdots 2n-1}{2\,6^n}$$

so ~~for~~ $x \in [3,5]$

$$|R_n(x)| \leq \frac{1\cdot 3\cdot 5\cdots 2n-1}{2(6^n)(n+1)!}\,|x-4|^{n+1} = \frac{1}{2\cdot 6^n}\left(\frac{1\cdot 3\cdot 5\cdots 2k-1\cdots 2n-1}{1\cdot 2\cdot 3\cdots k\cdots n\,(n+1)}\right)|x-4|^{n+1}$$

$$\leq \frac{2^n}{2\cdot 6^n}\cdot\frac{1}{n+1} = \frac{1}{(2n+2)3^n}$$

$\frac{2k-1}{k} \leq 2$

So, can estimate $\sqrt{x}$ on $[3,5]$ using degree $n=3$ polynomial.

$$\sqrt{x} \approx 2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2 + \frac{1}{512}(x-4)^3$$

$$|R_3(x)| \leq \frac{1}{8\cdot 27} = \frac{1}{216} < .0047$$

note, gets better and better closer x is to 4.

Evaluating limits:

$$\lim_{x\to 0} \frac{\sin x - x}{x^3} = ?$$ could use L'Hôpital's rule...

instead, use series:

$$\lim_{x\to 0} \frac{\sin x - x}{x^3} = \lim_{x\to 0} \frac{1}{x^3}\left(\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} - x\right)$$

$$= \lim_{x\to 0} \frac{1}{x^3} \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = \lim_{x\to 0} \underbrace{\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n-2}}_{\text{cont at } 0.}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)!} 0^{2n-2} = \frac{-1}{3!} = -\frac{1}{6}.$$

Can also multiply & divide power series:

Fact: product has rad of conv. = minimum of two radii

quotient has some radius of convergence (prov denom $\neq 0$ at cent)

– just like polynomials –

$$\left(\sum_{n=0}^{\infty} a_n x^n\right)\left(\sum_{n=0}^{\infty} b_n x^n\right) = \sum_{n=0}^{\infty}\left(\left(\sum_{k=0}^{n} a_k b_{n-k}\right)x^n\right)$$

e.g. $$\sin(x)\cos(x) = \left(\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}\right)\left(\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}\right)$$

$$= \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots\right)\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots\right)$$

$$= x - \left(\frac{1}{2} + \frac{1}{6}\right)x^3 + \left(\frac{1}{5!} + \frac{1}{3!2!} + \frac{1}{4!}\right)x^5 - \dots$$

[compare with $\cos x \sin x = \frac{\sin 2x}{2}$]