Math 231 4/6/11

Binomial series: Recall binomial expansion —

$(1+x) = 1+x$

$(1+x)^2 = 1+2x+x^2$

$(1+x)^3 = 1+3x+3x^2+x^3$

$(1+x)^4 = 1+4x+6x^2+4x^3+x^4$

$(1+x)^5 = 1+5x+10x^2+10x^3+5x^4+x^5$.

$$\begin{array}{c} 1 \\ 1 \quad 1 \\ 1 \quad 2 \quad 1 \\ 1 \quad 3 \quad 3 \quad 1 \\ 1 \quad 4 \quad 6 \quad 4 \quad 1 \\ 1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1 \end{array}$$

Pascal's triangle.

can also write, for any $k \in \mathbb{N}$

$$(1+x)^k = \sum_{n=0}^{k} \binom{k}{n} x^n$$

where $\binom{k}{n} = \frac{k!}{n!(n-k)!}$ = "k choose n" = the number of ways to choose n things from a set of k distinct things.

Eg: how many ways are there to choose 4 marbles from a set of 10 different colored marbles?
10 choices for 1st, 9 for second... so
$10 \cdot 9 \cdot 8 \cdot 7$ — but, it does matter what order we choose them, so how many repeats? $4 \cdot 3 \cdot 2 \cdot 1$ = # of ways to reorder them, so get

$$\frac{10 \cdot 9 \cdot 8 \cdot 7}{4 \cdot 3 \cdot 2 \cdot 1} = \frac{(10!/6!)}{4!} = \frac{10!}{6!\,4!}$$

What if $k = \frac{1}{2}$, what is $(1+x)^{\frac{1}{2}}$? What about any real value of k?

$f(x) = (1+x)^k$ find Maclaurin series:

$$f(x) = (1+x)^k$$
$$f'(x) = k(1+x)^{k-1}$$
$$f''(x) = k(k-1)(1+x)^{k-2}$$
$$\vdots$$
$$f^{(n)}(x) = k(k-1)\cdots(k-n+1)(1+x)^{k-n}$$
$$\vdots$$

Note: If $k \in \mathbb{N}$, then this is eventually 0 from some point on.

So

$$f^{(n)}(0) = k(k-1)\cdots(k-n+1)$$

define $\binom{k}{n} = \frac{k(k-1)\cdots(k-n+1)}{n!}$ $(=1, n=0)$

Note: agrees with old defn when $k \in \mathbb{N}$ and $n \leq k$.

then Maclaurin series is

$$\sum_{n=0}^{\infty} \binom{k}{n} x^n \qquad \text{radius of convergence?}$$

Ratio test:

$$\lim_{n \to \infty} \left| \frac{\binom{k}{n+1} x^{n+1}}{\binom{k}{n} x^n} \right| = \lim_{n \to \infty} \left| \frac{\frac{k(k-1)\cdots(k-n)}{(n+1)!}}{\frac{k(k-1)\cdots(k-n+1)}{n!}} \right| |x|$$

$$= |x| \lim_{n \to \infty} \frac{n-k}{n+1} = |x| < 1 \quad \text{so radius of conv.} = 1.$$

Q: $(1+x)^k \overset{?}{=} \sum_{n=0}^{\infty} \binom{k}{n} x^n$ for $|x|<1$?

A: Yes — can show $R_n(x) \to 0$ ... here's an easier trick.

Set $g(x) = \sum_{n=0}^{\infty} \binom{k}{n} x^n$ for $|x|<1$

compute: $g'(x) = \sum_{n=1}^{\infty} \binom{k}{n} n x^{n-1}$ and

$$g'(x)(1+x) = \sum_{n=1}^{\infty} \binom{k}{n} n x^{n-1} + \sum_{n=1}^{\infty} \binom{k}{n} n x^{n}$$

$$= \sum_{n=0}^{\infty} \binom{k}{n+1}(n+1) x^n + \sum_{n=0}^{\infty} \binom{k}{n} n x^n = \sum_{n=0}^{\infty} \left(\underbrace{\binom{k}{n+1}(n+1) + \binom{k}{n} n}_{} \right) x^n$$

$$= \sum_{n=0}^{\infty} k \binom{k}{n} x^n = k\, g(x)$$

$n \neq 0$

$$\frac{k(k-1)\ldots(k-n+1)(k-n)\cancel{(n+1)}}{(n+1)!} + \frac{k(k-1)\ldots(k-n+1)n}{n!}$$

$$= \frac{k(k-1)\ldots(k-n+1)(k-n) + k(k-1)\ldots(k-n+1)n}{n!}$$

$$= \frac{k(k-1)\ldots(k-n+1)}{n!}\big((k-n)+n\big)$$

$$= k\binom{k}{n}$$

so $g'(x) = \dfrac{k\,g(x)}{1+x}$ for $|x|<1$

now we show that $g(x) = (1+x)^k$ by showing $\dfrac{g(x)}{(1+x)^k} = h(x)$ is constant (and hence $=1$ since $g(0) = 1 = (1+0)^k$.)

It suffices to show $h'(x) = 0$. For this, we compute

$$h'(x) = \frac{(1+x)^k g'(x) - g(x)\, k (1+x)^{k-1}}{(1+x)^{2k}} = \frac{(1+x)^k \frac{k\,g(x)}{(1+x)} - g(x)\, k(1+x)^{k-1}}{(1+x)^{2k}} = 0 \quad ✓$$