Math 231 4/4/11

Recap: (1) Power Series $\sum_{n=0}^{\infty} c_n (x-a)^n$ converges for $x=a$ only or absolutely on $(a-R, a+R)$ for some $0 < R \leq \infty$.

R = radius of convergence.

on $(a-R, a+R)$ $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$ is differentiable w/ $f'(x) = \sum_{n=1}^{\infty} n c_n (x-a)^{n-1}$ and $\sum_{n=1}^{\infty} n c_n (x-a)^{n-1}$ has the same radius of convergence. Similarly

$\int f(x)\,dx = \sum_{n=0}^{\infty} \frac{c_n}{n+1} (x-a)^{n+1} + C$ and $\sum_{n=0}^{\infty} \frac{c_n}{n+1} (x-a)^{n+1}$ have the same radius of convergence.

(2) Taylor & Maclaurin series for a function $f(x)$:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n \qquad \text{(case } a=0 \text{ is Maclaurin series)}$$

CAUTION: NOT ALWAYS TRUE THAT $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$.

but if $f(x)$ is given by a power series in $(x-a)$, then that series is the Taylor series.

n^{th} Taylor polynomial is what we get by truncating:

$T_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!} (x-a)^i$ and remainder is $R_n(x) = f(x) - T_n(x)$.

$f(x)$ is given by a power series (hence Taylor series) on $(a-R, a+R)$ **if** $R_n(x) \to 0$ as $n \to \infty$ for all $x \in (a-R, a+R)$.

Main way to prove this is with.

Taylor's Inequality: if $|f^{(n+1)}(x)| \le M$ for $x \in (a-d, a+d)$, then

$$|R_n(x)| \le \frac{M}{(n+1)!}|x-a|^{n+1} \quad \text{for } x \in (a-d, a+d).$$

<u>Ex</u> $f(x) = e^x$, $a = 0$ (Maclaurin series).

$f^{(n)}(x) = e^x$ for every n and x, so Maclaurin series is

$$\sum_{n=0}^{\infty} \frac{e^0}{n!} x^n = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$ — has infinite radius of convergence by ratio test.

and remainder: fix <u>any</u> $d > 0$, let $x \in (-d, d)$

we have $|f^{(n+1)}(x)| = e^x \le e^d$ for every $x \in (-d, d)$ so

$$0 \le |R_n(x)| \le \frac{e^d}{(n+1)!}|x|^{n+1} \quad \text{and} \quad \lim_{n\to\infty} \frac{e^d |x|^{n+1}}{(n+1)!} = e^d \lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)!} = 0.$$

so, by squeeze thm, $\lim_{n\to\infty} R_n(x) = 0$, and

$$\boxed{e^x = \sum \frac{x^n}{n!}}$$

<u>EX</u> $f(x) = \sin(x)$, $a = 0$

$$f^{(n)}(x) = \begin{cases} \sin(x) & n = 4k \\ \cos(x) & n = 4k+1 \\ -\sin(x) & n = 4k+2 \\ -\cos(x) & n = 4k+3 \end{cases}, \quad k \in \mathbb{N}. \qquad f^{(n)}(0) = \begin{cases} 0 & n = 4k \\ 1 & n = 4k+1 = 2(2k)+1 \\ 0 & n = 4k+2 \\ -1 & n = 4k+3 = 2(2k+1)+1 \end{cases}$$

only <u>odd</u> terms are nonzero

$$\sum_{n=0}^{\infty} \frac{f^{(2n+1)}(0)}{(2n+1)!} x^{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$

[student]. Is $\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$?

Yes: $|f^{(n+1)}(x)| \le 1$ for every x, so

$0 \le |R_n(x)| \le \frac{1}{(n+1)!} |x|^{n+1}$ and $\lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)!} = 0$,

so squeeze thm applies again and $\lim_{n\to\infty} R_n(x) = 0$

$$\boxed{\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}}$$

EX Q. How do we see

$$\boxed{\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}} \; ?$$

A. differentiate:

$$\frac{d}{dx}(\sin(x)) = \frac{d}{dx}\left(\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} \right)$$

$$\cos x \qquad\qquad \sum_{n=0}^{\infty} (2n+1) \frac{(-1)^n}{(2n+1)!} x^{2n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$$

EX What is Maclaurin series for $\cos(2x)$? e^{x^2}?

$$\cos(2x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} (2x)^{2n} = \sum_{n=0}^{\infty} \frac{(-1)^n 4^n}{(2n)!} x^{2n} = \sum_{n=0}^{\infty} \frac{(-4)^n}{(2n)!} x^{2n}.$$

$$e^{x^2} = \sum_{n=0}^{\infty} \frac{(x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!}$$