Math 231 4/18/11

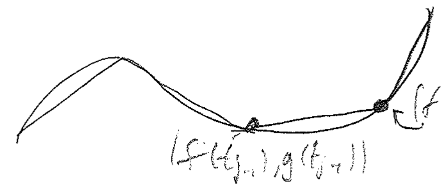
Arc length

$x = f(t)$, $y = g(t)$

$t \in [a, b]$

Approximate & take a limit:

$\Delta t = \frac{b-a}{n}$, $t_j = a + j\Delta t$



$$\text{Length} = \lim_{n\to\infty} \sum_{j=1}^{n} \sqrt{(f(t_j) - f(t_{j-1}))^2 + (g(t_j) - g(t_{j-1}))^2}$$

$$= \lim_{n\to\infty} \left(\sum_{j=1}^{n} \sqrt{\left(\frac{f(t_j) - f(t_{j-1})}{\Delta t}\right)^2 + \left(\frac{g(t_j) - g(t_{j-1})}{\Delta t}\right)^2} \right) \Delta t$$

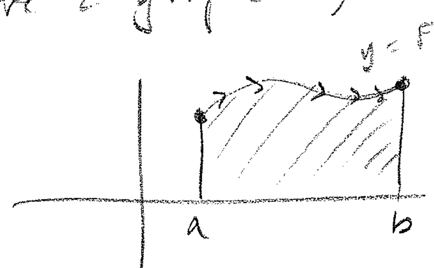
$$\underset{\text{with work}}{=} \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\, dt$$

$$\sqrt{(f'(t))^2 + (g'(t))^2} = \text{speed} = \lim_{h\to 0} \frac{\text{distance from } (f(t), g(t)) \text{ to } (f(t+h), g(t+h))}{h}$$

Areas

If $x = f(t)$ $y = g(t)$ is a parametrization of the graph of a function $y = F(x)$, then we can compute area between curve & graph by substitution: $(F > 0)$

$x = f(t)$, $y = g(t)$ $t \in [c, d]$



$g(t) = y = F(x) = F(f(t))$

so $g(t) = F(f(t))$ hence

$a = f(c)$

$b = f(d)$

$$\int_a^b F\,dx \underset{x = f(t)}{=} \int_c^d F(f(t)) f'(t)\, dt$$

$$= \int_c^d g(t) f'(t)\, dt.$$

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Surface Area

curve $x = f(t)$, $y = g(t)$, revolved around line L, $t \in [a, b]$

$$\int_a^b 2\pi (\text{dist to } L) \sqrt{(f')^2 + (g')^2}\, dt$$

[e.g. around x-axis: $\int_a^b 2\pi |g(t)| \sqrt{f'^2 + g'^2}\, dt.$]

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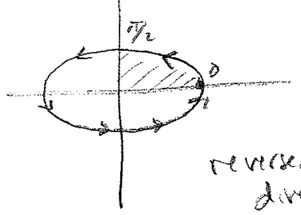
Ex Area enclosed by an ellipse $x = a\cos\theta$, $y = b\sin\theta$, $0 \le \theta \le 2\pi$. $a, b > 0$.



reversed direction

$$-4\int_0^{\pi/2} b\sin\theta\,(-a\sin\theta)\,d\theta$$

$$= 4ab\int_0^{\pi/2} \sin^2\theta\,d\theta$$

$$= 4ab\int_0^{\pi/2} \left(\tfrac{1}{2}(1-\cos 2\theta)\right) d\theta$$

$$= 2ab\int_0^{\pi/2} 1-\cos 2\theta\,d\theta = 2ab\left[\left(\theta - \frac{\sin 2\theta}{2}\right)\Big|_0^{\pi/2}\right]$$

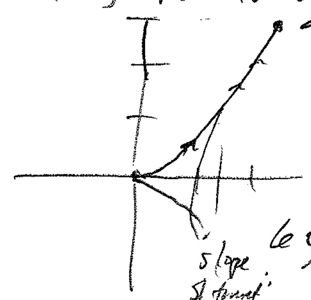
$$= 2ab\left(\tfrac{\pi}{2}\right) = \pi ab$$

Integral for Length? $\displaystyle\int_0^{2\pi}\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}\,d\theta$ (not an integral we can compute)

Ex Surface area for surface obtained by revolving

$x = 3t - t^3$, $y = 3t^2$ $\quad 0 \le t \le 1$

around x-axis?



slope of tangent: $\dfrac{6t}{3-3t^2} = \dfrac{2t}{1-t^2}$

$y = 3t^2 \ge 0$ for $0 \le t \le 1$

$$\text{Surface Area} = \int_0^1 2\pi(3t^2)\sqrt{(3-3t^2)^2+(6t)^2}\,dt = 6\pi\int_0^1 t^2\sqrt{9+18t^2+9t^4}\,dt$$

$$= 6\pi\int_0^1 3t^2\sqrt{t^4+2t^2+1}\,dt = 18\pi\int_0^1 t^2(1+t^2)\,dt = 18\pi\left[\frac{t^3}{3}+\frac{t^5}{5}\right]_0^1$$

$$= 18\pi\left(\frac{1}{3}+\frac{1}{5}\right) = \frac{18\pi\cdot 8}{15} = \frac{48\pi}{5}$$