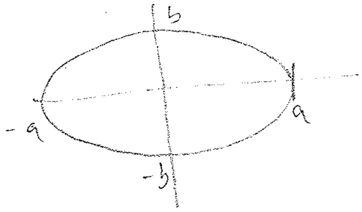
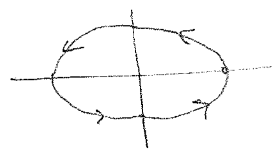
Ex [student] Find a parametrization of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ traversed clockwise once.



First find a parametrization

$x = a\cos t$ $y = b\sin t$ has

$\frac{x^2}{a^2}+\frac{y^2}{b^2} = \frac{a^2\cos^2 t}{a^2}+\frac{b^2\sin^2 t}{b^2} = \cos^2 t+\sin^2 t = 1$.



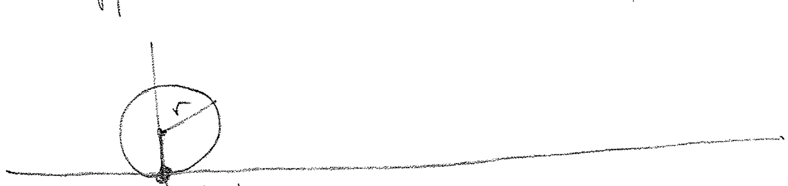
to reverse the direction, can change the sign of t (and choose domain accordingly)

$x = a\cos(-t)$ $y = b\sin(-t)$ $t \in [0, 2\pi]$.
$= a\cos(t)$ $= -b\sin(t)$

To find parametrizations in more complicated settings, it helps to break the problem up into pieces.

cycloid — parametric curve describing a particle on a rolling wheel.

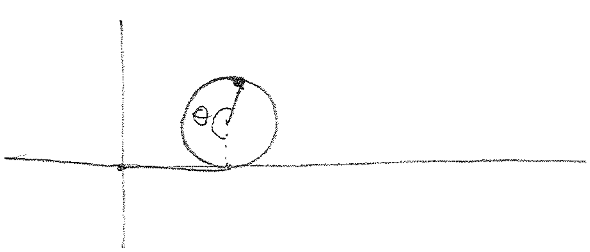
Suppose circle (wheel) has radius $r > 0$, fix a starting point, say



Start.

as wheel rolls, distance t say, how has angle changed? Answer: relation between t & θ is

$r\theta = t$. — distance traveled is length of arc on circle.



From this, we can find position on circle using t as a parameter:

for a circle of radius r centered at 0, traversed clockwise, get:

$x = r\cos(\theta)$ $y = -r\sin(\theta)$. Starting at the "bottom" of circle means

add $\pi/2$ to θ to get

$x = r\cos(\theta + \pi/2)$ $y = -r\sin(\theta + \pi/2)$

$= -r\sin(\theta)$ $= -r\cos(\theta)$

In terms of t then

$x = -r\sin(\frac{t}{r})$ $y = -r\cos(\frac{t}{r})$

Now we just need to change the center:

at time t, the center is at (t, r). So, get

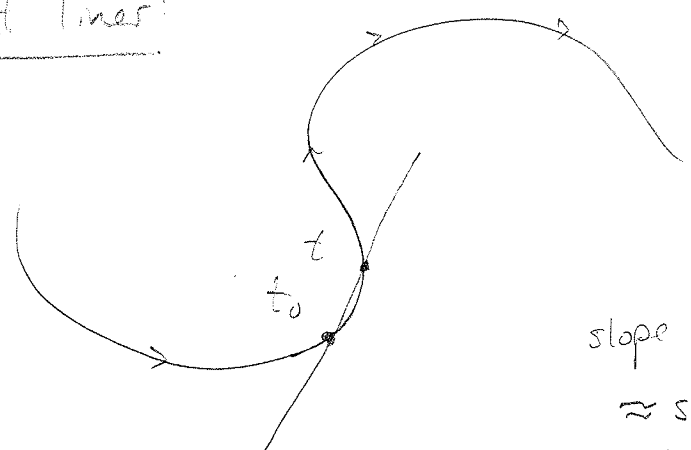
$x = t - r\sin(\frac{t}{r})$ $y = r - r\cos(\frac{t}{r})$

Could also have left it in terms of θ (i.e. use θ as parameter)

$x = r\theta - r\sin\theta$ $y = r - r\cos(\theta)$

We can use calculus to provide more information about a parameterized curve:

<u>tangent lines</u>:



$x = f(t) \quad y = g(t)$

slope of tangent line at t_0?
$\approx$ slope of secant line through
$(f(t_0), g(t_0))$ & $(f(t), g(t))$
$= \dfrac{g(t)-g(t_0)}{f(t)-f(t_0)}$

take limit to get exact slope:

$$\lim_{t\to t_0} \frac{g(t)-g(t_0)}{f(t)-f(t_0)} = \lim_{t\to t_0} \frac{(g(t)-g(t_0))/(t-t_0)}{(f(t)-f(t_0))/(t-t_0)} = \frac{g'(t_0)}{f'(t_0)}$$

provided $f'(t_0), g'(t_0)$ not both 0.

<u>Ex</u> Where is the tangent line to

$$x = 2t^3+3t^2-12t \quad y = 2t^3+3t^2+1$$

horizontal? vertical? What is equation of tangent line at time $t=2$

$x' = 6t^2+6t-12$
$= 6(t^2+t-2)$
$= 6(t+2)(t-1)$

$y' = 6t^2+6t$
$= 6t(t+1)$

horiz: $t = 0, -1$
vert: $t = 1, -2$

Slope at $t=2$: $\frac{36}{24} = \frac{3}{2}$ position at $t=2$:

$x = 16+12-24 = 4$
$y = 16+12+1 = 29$

$y - 29 = \frac{3}{2}(x-4)$
or
$y = \frac{3}{2}x + 23$.