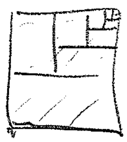
Last time we saw that by rearranging the terms of the alternating harmonic series $\sum_{n=1}^{\infty} (-1)^n/n$, we could get any sum we wanted. — not a contradiction since sum of series is taken in specified order.

On the other hand, $\sum_{n=1}^{\infty} \frac{1}{2^n}$ always gives 1, no matter how the terms are rearranged. — clear geometrically: all areas fit in square



so a sum of any finite # of terms is ≤ 1. On the other hand, eventually get all rectangles, so finite sums become arbitrarily close to 1

difference: all terms of $\sum \frac{1}{2^n}$ are positive but more importantly, in $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$, sum of pos. terms & sum of negative terms diverge.

Defn If the sum of the pos. terms & sum of neg. terms of $\sum_{n=1}^{\infty} a_n$ converge, say that $\sum_{n=1}^{\infty} a_n$ converges absolutely. Equivalently, $\sum_{n=1}^{\infty} a_n$ converges absolutely iff $\sum_{n=1}^{\infty} |a_n|$ converges. If $\sum_{n=1}^{\infty} a_n$ converges, but $\sum_{n=1}^{\infty} |a_n|$ diverges, say $\sum a_n$ converges conditionally.

So, $\sum_{n=1}^{\infty} \frac{1}{2^n} = \sum_{n=1}^{\infty} \left|\frac{1}{2^n}\right| \Longrightarrow \sum_{n=1}^{\infty} \frac{1}{2^n}$ converges absolutely

but $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges conditionally since (a) it converges and (b) $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

* Need both for conditional convergence. *

Fact: If $\sum a_n$ converges conditionally, it can be made to converge to any number by rearranging the terms (as was done for $\sum \frac{(-1)^n}{n}$)

<u>Thrm</u> absolute convergence implies convergence and any rearrangement has same sum.

to see this, observe that $0 \le a_n + |a_n| \le 2|a_n| \; \forall n$ so

$\sum |a_n|$ converges $\Rightarrow$ $\sum 2|a_n|$ converges $\Rightarrow$ $\sum a_n + |a_n|$ converges $\Rightarrow$ $\sum a_n = \sum (a_n + |a_n|) - \sum |a_n|$ converges

this implies sum of pos & neg. terms converge,

If S_n, P_n and M_n are the partial sum of 1st n terms, P_n the sum of those pos. and M_n the sum of those negative, then

$S_n = P_n + M_n$ and $S = \lim_{n\to\infty} S_n = \lim_{n\to\infty} P_n + \lim_{n\to\infty} M_n$.

rearranging and summing either positive or negative terms does not affect sum — compare w/ $\sum_{n=1}^{\infty} \frac{1}{2^n}$, so sum S is same for any rearrangement.

<u>EX</u> $\sum \frac{\cos(n)}{n^2}$ converge or diverge? abs or cond?

$\sum \frac{|\cos(n)|}{n^2}$ has $\frac{|\cos(n)|}{n^2} \le \frac{1}{n^2}$, & $\sum \frac{1}{n^2}$ converges.

So $\sum \frac{\cos n}{n^2}$ converges absolutely (hence it converges).

<u>EX</u> $\sum \frac{(-1)^n}{n}$ converges conditionally — already saw it

$\sum \frac{(-1)^n}{n^{3/2}}$? converges by alt. series test.

but also converges abs. since $\sum \left|\frac{(-1)^n}{n^{3/2}}\right| = \sum \frac{1}{n^{3/2}}$ and $3/2 > 1$.

convergence by alt. series test does <u>not</u> imply conditional convergence

EX [student] $\sum_{n=2}^{\infty} \frac{(-1)^n}{n\ln(n)}$ converge / diverge? abs or cond?

$\sum_{n=2}^{\infty} \frac{(-1)^n}{n\ln(n)}$ converges by alt. series test since
- $\frac{1}{n\ln(n)} > 0$
- $\frac{1}{(n+1)\ln(n+1)} < \frac{1}{n\ln(n)}$ $\forall n$.
- $\lim_{n\to\infty} \frac{1}{n\ln(n)} = 0$.

Since $f(x) = \frac{1}{x\ln x}$ pos. decreasing on $[2,\infty)$ and

$$\int_2^{\infty} \frac{dx}{x\ln x} = \lim_{t\to\infty} \int_2^t \frac{dx}{x\ln x} = \lim_{t\to\infty} \int_{\ln 2}^{\ln t} \frac{du}{u} = \lim_{t\to\infty} \ln(\ln(t)) - \ln(\ln(2)) = \infty.$$

$u = \ln x$
$du = \frac{1}{x}dx$

implies $\sum \frac{1}{n\ln n}$ converges by integral test.

The comparison test gives two more useful tests.

$\boxed{\text{Ratio Test}}$ Consider series $\sum_{n=1}^{\infty} a_n$ and suppose $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L$

Then (1) If $L < 1$, $\sum a_n$ converges absolutely

(2) if $L > 1$ (or if $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \infty$) then $\sum a_n$ diverges.

(3) if $L = 1$ — test is inconclusive.

Why does this work?

Suppose (1): eventually $\left|\frac{a_{n+1}}{a_n}\right| < r$ for some $L < r < 1$.

That is for some $N > 0$ we have $\left|\frac{a_{n+1}}{a_n}\right| < r$ when $n > N$.

for $k>0$, $\left|\frac{a_{n+k}}{a_{n+k-1}}\right| < r$ so

Then $|a_{n+k}| < r|a_{n+k-1}| < r^2|a_{n+k-2}| < \ldots < r^k|a_n|$

Consider the "tail" of the series: $\sum_{k=0}^{\infty} |a_{n+k}|$

note $\sum_{n=1}^{\infty} |a_n|$ converges if and only if this tail does, and the tail is bounded, term by term, by $\sum_{k=0}^{\infty} |a_n| r^k$

a constant.

$0<r<1 \Rightarrow$ series converges, so conv. absolutely.

for 2, note $|a_{n+1}|>|a_n|$ eventually $\Rightarrow \lim_{n\to\infty} |a_n| \neq 0 \Rightarrow \lim_{n\to\infty} a_n \neq 0$, diverges by n^{th} term test.

Ex $\sum \frac{(-1)^n n\ln(n)}{2^n}$ last time a little messy - easy now:

$$\lim_{n\to\infty}\left|\frac{(-1)^{n+1}(n+1)\ln(n+1)}{2^{n+1}} \bigg/ \frac{(-1)^n n\ln(n)}{2^n}\right| = \lim_{n\to\infty} \frac{n+1}{n}\,\frac{\ln(n+1)}{\ln(n)}\cdot\frac{1}{2} = \frac{1}{2} < 1$$

($\frac{n+1}{n} \to 1$, $\frac{\ln(n+1)}{\ln(n)} \to 1$)

converges absolutely.

Ex [student?] $\sum \frac{n^n n!}{(2n)!}$?

$$\lim_{n\to\infty} \frac{(n+1)^{n+1}(n+1)!}{(2(n+1))!} \bigg/ \frac{n^n n!}{(2n)!} = \lim_{n\to\infty} \frac{(n+1)^{n+1}}{n^n}\cdot\frac{(n+1)!}{n!}\cdot\frac{(2n)!}{(2n+2)!}$$

$$= \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n (n+1)^2 \cdot \frac{1}{(2n+2)(2n+1)} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n \frac{n^2+2n+1}{4n^2+6n+2} = \frac{e}{4} < 1.$$

$$\lim_{x\to\infty} x\ln\left(1+\frac{1}{x}\right) = \lim_{x\to\infty} \frac{\ln(1+\frac{1}{x})}{\frac{1}{x}} = \lim_{x\to\infty} \frac{\frac{1}{1+\frac{1}{x}}\cdot\frac{-1}{x^2}}{\frac{-1}{x^2}} = 1$$