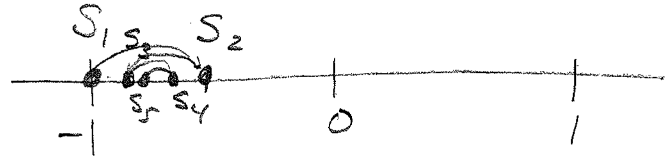
Math 231 3/4/11

Alternating series

Ex $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ (alternating harmonic series)

converges:



$S_1 = -1$

$S_2 = -1 + \frac{1}{2}$

$S_3 = -1 + \frac{1}{2} - \frac{1}{3}$

$S_4 = -1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4}$

$S_5 = -1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5}$

$\{S_n\}$ jumps back and forth a smaller & smaller amount (an amount tends to 0) never escaping past the previous terms.

We can extract the idea to give

Alternating series test

Suppose $\displaystyle\sum_{n=1}^{\infty} (-1)^n b_n$ w/
1. $b_n > 0$
2. $b_{n+1} \leq b_n$ for all n
3. $\displaystyle\lim_{n\to\infty} b_n = 0$.

Then series converges.

Basically the idea above —

observe that even terms are decreasing — clear from picture

$$S_{2n} = S_{2n-2} - b_{2n-1} + b_{2n} = S_{2(n-1)} + \underbrace{(b_{2n} - b_{2n-1})}_{\geq 0} \geq S_{2(n-1)}$$

so $S_2 \geq S_4 \geq S_6 \geq \ldots$

& bounded below (also clear from picture)

$$S_{2n} = -b_1 + \underbrace{(b_2 - b_3)}_{\geq 0} + \underbrace{(b_4 - b_5)}_{\geq 0} + \ldots + \underbrace{(b_{2n-2} - b_{2n-1})}_{\geq 0} + \underbrace{b_{2n}}_{\geq 0} \geq -b_1$$

bound monotone sequence, so converges

$\lim_{n\to\infty} S_{2n} = S$.

Now show that odd terms also converge to S:

$$\lim_{n\to\infty} S_{2n+1} = \lim_{n\to\infty}\left(S_{2n} - b_{2n+1}\right) = \lim_{n\to\infty} S_{2n} - \lim_{n\to\infty} b_{2n+1} = S - 0 = S$$

If even terms & odd terms both converge to same limit, then entire sequence converges: $\lim_{n\to\infty} S_n = S$.

Ex (students) converge or diverge - what test?

① $\sum_{n=1}^{\infty} (-1)^n e^{-n}$

conv. by alt series test.

since • $e^{-n} > e^{-(n+1)} > 0$

• $\lim_{n\to\infty} e^{-n} = 0$

or geometric series

$= \sum_{n=1}^{\infty} \left(-\frac{1}{e}\right)^n$

② $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{\sqrt{n}}$

• $\frac{1}{\sqrt{n}} \geq \frac{1}{\sqrt{n+1}} \geq 0$

• $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$

$\Rightarrow$ converge by alt series test.

③ $\sum_{n=1}^{\infty} (-1)^n \left(1+\frac{1}{n}\right)^n$

$\lim_{n\to\infty} (-1)^n \left(1+\frac{1}{n}\right)^n \neq 0$ so diverges by n^{th} term test. - terms don't limit to 0.

[alt series test does not give divergence].

Rearranging the alternating harmonic series:

We know $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, and $S_n = \sum_{i=1}^{n} \frac{1}{i}$ has $\lim_{n\to\infty} S_n = \infty$

Same is true if we add up only even or odd terms:

$t_n = \sum_{i=1}^{n} \frac{1}{2i}$ $\quad u_n = \sum_{i=1}^{n} \frac{1}{2i-1}$

$$t_n = \frac{1}{2}\sum_{i=1}^{n} \frac{1}{i} = \frac{1}{2} s_n$$

$$u_n = \sum_{i=1}^{n} \frac{1}{2i-1} \geq \sum_{i=1}^{n} \frac{1}{2i} = \frac{1}{2} s_n$$

So $\lim\limits_{n\to\infty} u_n = \lim\limits_{n\to\infty} t_n = \lim\limits_{n\to\infty} s_n = \infty$.

but $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n} = \lim_{n\to\infty} \underbrace{(t_n - u_n)}_{\| \sum_{i=1}^{2n} \frac{(-1)^i}{i}}$ converges.

<u>Q</u>: What happens if we change the order in which we add the terms?

<u>A</u>: We can get something different... In fact, we can get <u>anything</u>!

<u>Why?</u> Pick <u>any</u> real # t, say $t > 0$ for convenience

add positive terms (in order) just until sum is $\geq t$.

$\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \dots + \frac{1}{2n_1} \geq t$, then add 1st negative term, drops below t, then add +ve terms to make it $\geq t$, add next neg. term, add more +ve, add next negative... continue.

we eventually use <u>every</u> even & odd term, and deviate less & less from t. A similar idea gets sum $< t$.

This is why we're very careful in defining sums of series as limits of sequence of partial sums — this is very much dependent (as this ex shows) on <u>order</u> of addition. Next week will see that this weird behavior can be avoided in most nice situations.