Math231 3/2/11

<u>Ex</u> [students] does $\sum_{n=1}^{\infty} \frac{1}{n(\ln n)}$ converge or diverge?

$$\int_2^{\infty} \frac{dx}{x\ln(x)} = \lim_{t\to\infty} \int_2^t \frac{dx}{x\ln x} = \lim_{t\to\infty} \int_{\ln(2)}^{\ln(t)} \frac{du}{u} = \lim_{t\to\infty} \ln|u|\Big|_{\ln 2}^{\ln(t)} = \lim_{t\to\infty} \ln(\ln(t)) - \ln(\ln(2)) = \infty$$

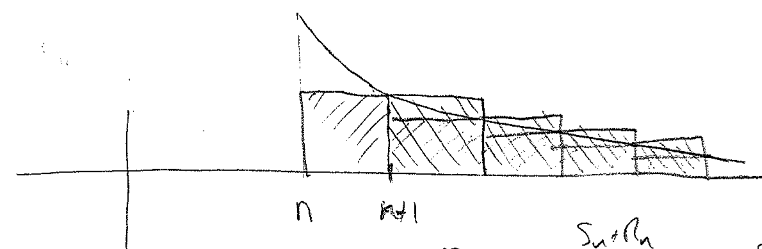
$u = \ln x$
$du = \frac{1}{x}dx$

so diverges by integral test

can start anywhere — first few (several) terms are unimportant.

Might not be able to compute $\sum_{n=1}^{\infty} a_n$ explicitly, but integral test can tell us convergence — In this case, how close is s_n to $s = \sum_{n=1}^{\infty} a_n$? That is, how large can remainder be, $R_n = s - s_n$?

$$R_n = \sum_{i=n+1}^{\infty} a_i$$



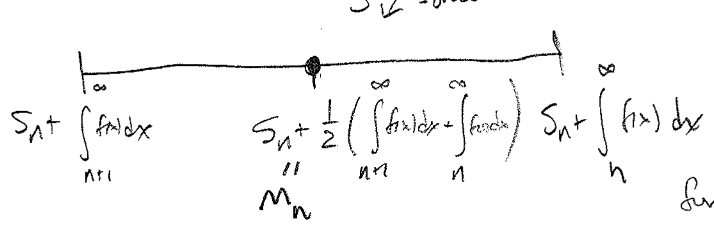
$$\int_{n+1}^{\infty} f(x)\,dx \le R_n \le \int_n^{\infty} f(x)\,dx \implies s_n + \int_{n+1}^{\infty} f(x)\,dx \le \overset{s_n + R_n}{\overset{\shortparallel}{s}} \le s_n + \int_n^{\infty} f(x)\,dx$$

<u>Ex</u> $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, how large must n be so $R_n \le .0001 = \frac{1}{10000}$?

$$R_n \le \int_n^{\infty} \frac{1}{x^2}\,dx = \lim_{t\to\infty} \int_n^t \frac{1}{x^2}\,dx = \lim_{t\to\infty} \frac{-1}{x}\Big|_n^t = \lim_{t\to\infty}\left(\frac{1}{n} - \frac{1}{t}\right) = \frac{1}{n}.$$

so, can take $n \ge 1000$, then $\frac{1}{n} \le \frac{1}{1000}$.

Can also estimate using midpoint of interval: s is somewhere.



Error $= |s - M_n|$

$\frac{1}{2}$ length of interval

$$\le \frac{1}{2}\left(\int_n^{\infty} f(x)\,dx - \int_{n+1}^{\infty} f(x)\,dx\right) = \frac{1}{2}\int_n^{n+1} f(x)\,dx$$

better!

$n \ge 71$

for $\frac{1}{10000}$ need only $\frac{1}{2}\left(\frac{1}{n} - \frac{1}{n+1}\right) = \frac{1}{2}\frac{1}{n(n+1)} \le \frac{1}{10000}$

The comparison test

[compare with statement of comparison test for improper integrals]

$\sum a_n$, $\sum b_n$ w/ $0 \le a_n \le b_n$ (for sufficiently large n)

(1) If $\sum b_n$ converges, so does $\sum a_n$

(2) If $\sum a_n$ diverges, so does $\sum b_n$.

– clear: $t_n = \sum_{i=1}^{n} b_i$, $s_n = \sum_{i=1}^{n} a_i$ so $s_n \le t_n$

If $\lim_{n\to\infty} t_n$ exists, s_n is bounded monotone hence convergent.

If $\lim_{n\to\infty} s_n$ diverges, then $\lim_{n\to\infty} s_n = \infty$ since it's monotone, so $\lim_{n\to\infty} t_n = \infty$

__Ex__ $\sum_{n=1}^{\infty} \frac{\sin^2 n}{2^n+3}$

since $0 \le \frac{\sin^2 n}{2^n+3} \le \frac{1}{2^n}$ & $\sum_{n=1}^{\infty} \frac{1}{2^n}$ converges, so $\sum_{n=1}^{\infty} \frac{\sin^2 n}{2^n+3}$ converges by comparison test

What do we compare with?

- $\sum_{n=1}^{\infty} \frac{1}{n^p}$ – p series, converges iff $p > 1$ by integral test
- geometric series

__Ex__ (student) $\sum_{n=1}^{\infty} \frac{n-1}{n^2\sqrt{n}}$

$\frac{n-1}{n^2\sqrt{n}} = \frac{n-1}{n} \cdot \frac{1}{n^{3/2}} \le \frac{1}{n^{3/2}}$. $\sum \frac{1}{n^{3/2}}$ converges, so $\sum \frac{n-1}{n^2\sqrt{n}}$ converges by comparison test

Ex (Students) $\sum_{n=1}^{\infty} \frac{\ln(n^{2n})}{n^2}$

$$\frac{\ln(n^{2n})}{n^2} = \frac{2n\ln(n)}{n^2} = \frac{2\ln(n)}{n} > \frac{2}{n} \quad \text{when } n \geq 3$$

by comparison test, $\sum_{n=1}^{\infty} \frac{\ln(n^{2n})}{n^2}$ diverges since $\sum \frac{2}{n}$ diverges.

[could also used integral test & comparison test for integrals]

Ex $\sum_{n=1}^{\infty} \frac{n!}{n^n}$

$$\frac{n!}{n^n} = \frac{1\cdot 2\cdot \cdots \cdot n}{n\cdot n \cdots n} \leq \frac{2}{n^2}\cdot\frac{3}{n}\cdot\frac{4}{n}\cdots\frac{n}{n} \leq \frac{2}{n^2}.$$ since $\sum \frac{2}{n^2}$ converges,

so does $\sum \frac{n!}{n^n}$ by comparison test,

some times exact comparisons are difficult, even when its clear what we should do:

$$\sum_1 \frac{n^2\ln(n)}{2^n}$$

Could observe that $\lim_{n\to\infty} \frac{n^2\ln(n)}{(\sqrt{2})^n} = 0$ and $\sum \frac{1}{(\sqrt{2})^n}$ converges, so

for larger n, $\frac{n^2\ln(n)}{2^n} = \frac{1}{(\sqrt{2})^n}\frac{n^2\ln(n)}{(\sqrt{2})^n} \leq \frac{1}{(\sqrt{2})^n}$

Simpler to apply:

$\boxed{\text{limit comparison test}}$

- If $a_n, b_n > 0$ and $\lim_{n\to\infty} \frac{a_n}{b_n} = c$, then if $0<c<\infty$ $\sum a_n$, $\sum b_n$ both converge or both diverge.
- If $c = 0$, and $\sum b_n$ converges, then $\sum a_n$ converges
- If $c = \infty$ and $\sum b_n$ diverges then $\sum a_n$ diverges

Ex: $\displaystyle\sum_{n=1}^{\infty} \frac{n+5}{\sqrt[3]{n^7+n^2}}$

$0 \le \dfrac{n+5}{\sqrt[3]{n^7+n^2}}$ $\sim \dfrac{n}{n^{7/3}}$ compare w/ $\dfrac{1}{n^{4/3}} \ge 0$

$$\lim_{n\to\infty} \frac{n+5}{\sqrt[3]{n^7+n^2}} \Big/ \frac{1}{n^{4/3}} = \lim_{n\to\infty} \frac{n^{7/3}+5n^{4/3}}{\sqrt[3]{n^7+n^2}} = \lim_{n\to\infty} \frac{1+5/n^{-1}}{\sqrt{1+1/n^5}} = 1$$

since $\sum \frac{1}{n^{4/3}}$ converges, so does $\sum \frac{n+5}{\sqrt[3]{n^7+n^2}}$ by limit comparison test.

Ex: (student) $\displaystyle\sum_{n=1}^{\infty} \frac{\sin(1/n)}{n}$ $\qquad \dfrac{\sin(1/n)}{n} > 0$, $\sin(1/n) \xrightarrow{\text{like}} 1/n$.

$$\lim_{n\to\infty} \left(\frac{\sin(1/n)}{n}\right) \Big/ (1/n^2) = \lim_{n\to\infty} \frac{\sin(1/n)}{1/n} = 1$$

since $\lim_{n\to\infty} \frac{\sin x}{x} = 1$