Math 231 3-14-11

Power series

We can think of the geometric series as a "series with a variable" and write

$$\sum_{n=1}^{\infty} ax^n$$

This defines a function whose domain is set of real # x for which the series converges. The value is the sum.

Q What is the domain? A. $(-1,1)$.

Of course, we can write down the function in another form:

$$\sum_{n=0}^{\infty} ax^n = \frac{a}{1-x}.$$

This is a special case of

Defn A power series (or power series in x) is an expression

$$\sum_{n=0}^{\infty} c_n x^n$$

where $\{c_n\}_{n=0}^{\infty}$ is a sequence of real numbers.

Writing this out we have

$$c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \ldots$$

So, this looks like a polynomial with infinitely many terms.

We will see that it also behaves like one where it converges.

We'll also consider

Defn a power series in $(x-a)$ is an expression $\sum_{n=0}^{\infty} c_n (x-a)^n$ w/ $\{c_n\}_{n=0}^{\infty}$ a sequence of real #'s

<u>Theorem</u> For $\sum_{n=0}^{\infty} c_n(x-a)^n$, exactly one of the following holds.

(1) series converges for $x=a$ only

(2) series converges absolutely ~~for~~ every real # x.

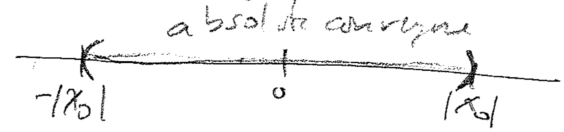
(3) There is an $R>0$ so that the series

- converges absolutely when $|x-a|<R$
- diverges when $|x-a|>R$

To see why, suppose for simplicity that $a=0$, so we have

$$\sum_{n=0}^{\infty} c_n x^n$$

Basically ~~we~~ need to check that if $\sum_{n=0}^{\infty} c_n x_0^n$ converges for some number $x_0 \neq 0$, then if x is any number with $|x|<|x_0|$, then $\sum c_n x^n$ converges.



Since $\sum c_n x_0^n$ converges, $\lim_{n\to\infty} c_n x_0^n = 0 \Rightarrow \lim_{n\to\infty} |c_n x_0^n| = 0$.

So, this means that $\{|c_n x_0^n|\}$ is bounded by some number $C>0$:

$$|c_n x_0^n| < C$$

Now apply comparison test to $\sum |c_n x^n|$, comparing with convergent geometric series $\sum C\left|\frac{x}{x_0}\right|^n$:

$$|c_n x^n| = \left|c_n x_0^n \left(\frac{x}{x_0}\right)^n\right| \leq C\left|\frac{x}{x_0}\right|^n$$

The "number" $\underset{(1)}{0} \le R \le \underset{(3)}{\infty}$ from this theorem is called the <u>radius of convergence</u> of the power series.

<u>Ex</u> ① Geometric series $\sum_{n=0}^{\infty} a x^n$ has radius of converg. $= 1$.

② $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ <u>Ratio test</u> $\lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right| = \lim_{n\to\infty} \frac{|x|}{n+1} = 0 < 1$

so series converges absolutely for all $x \in \mathbb{R}$, thus the series has <u>infinite radius of convergence</u>.

③ $\sum_{n=0}^{\infty} \frac{(-1)^n n^n x^n}{n!}$ <u>Ratio test</u>: $\lim_{n\to\infty} \frac{((n+1)|x|)^{n+1}}{(n+1)!} \cdot \frac{n!}{n^n |x|^n}$

$= \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^n |x| = e|x| < 1$

Converges absolutely for $|x| < \frac{1}{e}$, rad. of conv. $= \frac{1}{e}$

diverges for $|x| > \frac{1}{e}$.

④ $\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n^{1000}}$ Ratio test $\lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)^{1000}} \cdot \frac{n^{1000}}{|x|^n} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^{1000} |x| = |x| < 1$

rad. of convergence $= 1$

⑤ $\sum_{n=0}^{\infty} n^n x^n$ <u>root test</u>: $\lim_{n\to\infty} \sqrt[n]{|n^n x^n|} = \lim_{n\to\infty} n|x| = \infty$.

rad. of convergence $= 0$.