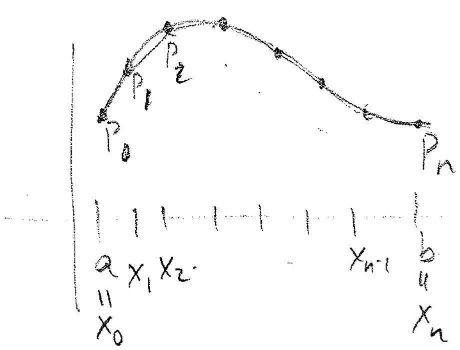
Math 231 2/9/11

Arc length

What is the length of a curve?

Imagine a tiny bug is walking along the curve. The total distance travelled is the length. We can estimate the length by picking points along the curve, adding up the distance between consecutive points:

$$\text{length} \approx |\overline{P_0P_1}| + |\overline{P_1P_2}| + \dots\dots + |\overline{P_{n-1}P_n}|$$



Suppose curve is graph of a function $y=f(x)$ on $[a,b]$. $P_i = (x_i, f(x_i))$

$|\overline{P_{i-1}P_i}|$ = distance from $(x_{i-1}, f(x_{i-1}))$ to $(x_i, f(x_i))$

$$= \sqrt{(x_i - x_{i-1})^2 + (f(x_i) - f(x_{i-1}))^2}$$

$a = x_0 < x_1 < \dots < x_{n-1} < x_n = b$, $x_i = x_{i-1} + \Delta x$ $= \sqrt{1 + \left(\dfrac{f(x_i) - f(x_{i-1})}{\Delta x}\right)^2}\ \Delta x$

$\Delta x = \frac{b-a}{n}$

$$\text{length} = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{1 + \left(\frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}}\right)^2}\ \Delta x$$

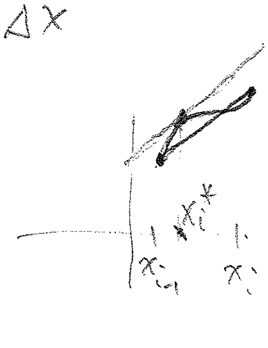
$$= \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{1 + (f'(x_i^*))^2}\ \Delta x$$

$$= \int_a^b \sqrt{1 + (f'(x))^2}\, dx$$

Mean Value theorem

There is $x_{i-1} \le x_i^* \le x_i$ w/ so that

$$\frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}} = f'(x_i^*)$$



If f' continuous on $[a,b]$ then length of graph over this interval is

$$\int_a^b \sqrt{1+(f'(x))^2}\,dx$$

EX. Length of graph of $f(x) = 2\sqrt{(x+4)^3}$ when $0 \le x \le 2$

$$f'(x) = \frac{3(x+4)^2}{\sqrt{(x+4)^3}} = 3\sqrt{x+4} \quad \text{so}$$

$$\text{length} = \int_0^2 \sqrt{1+9(x+4)}\,dx = \frac{1}{9}\int_{37}^{55} \sqrt{u}\,du$$

$u = 9x+37$

$du = 9dx$

$$= \frac{1}{9}\cdot\frac{2}{3}\left(55^{3/2} - 37^{3/2}\right) = \frac{2}{27}\left(55^{3/2} - 37^{3/2}\right)$$

EX [Students] Length of graph $f(x) = \frac{x^4}{8} + \frac{1}{4x^2}$ when $1 \le x \le 2$

$$f'(x) = \frac{x^3}{2} - \frac{1}{2x^3} = \frac{1}{2}(x^3 - x^{-3})$$

$$\int_1^2 \sqrt{1+\tfrac{1}{4}(x^3-x^{-3})^2}\,dx = \int_1^2 \sqrt{1+\tfrac{1}{4}x^6 - \tfrac{1}{2} + \tfrac{1}{4}x^{-6}}\,dx = \frac{1}{2}\int_1^2 \sqrt{x^6+2+x^{-6}}\,dx$$

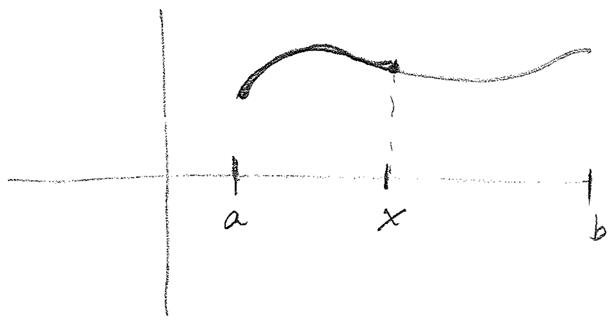
$$= \frac{1}{2}\int_1^2 \sqrt{(x^3+x^{-3})^2}\,dx = \frac{1}{2}\int_1^2 x^3+x^{-3}\,dx = \frac{1}{2}\left(\frac{x^4}{4} - \frac{1}{2}x^{-2}\right)\Big|_1^2$$

$$= \frac{1}{8}\left(16 - \frac{1}{2} - 1 + \frac{1}{2}\right) = \frac{15}{8}$$

– lucky: in general, can't compute explicitly.

[may be asked to "set up integral"]

As bug walks along the curve, we can ask at any point, how far it has walked — given $x \in [a,b]$, what is the length from a to x:



$$s(x) = \int_a^x \sqrt{1+(f'(t))^2}\, dt$$

$s(x)$ = arc length function.

fundamental theorem of calculus:

$$s'(x) = \sqrt{1+(f'(x))^2}$$

Q. What does this represent?

A. $s'(x) = \lim_{h \to 0} \frac{s(x+h)-s(x)}{h}$ = rate of change of distance traveled
= speed.

so $\sqrt{1+(f'(x))^2}$ is speed. Integrating speed gives distance traveled, i.e. arc length.

Ex [student] Find the arc length function for the graph of $f(x) = \ln(\cos(x))$ starting at $x=0$, as a function defined on $[0, \pi/2)$

$$f'(x) = \frac{-\sin(x)}{\cos(x)} = -\tan(x)$$

$$s(x) = \int_0^x \sqrt{1+\tan^2 t}\, dt = \int_0^x \sec x\, dx = \ln|\sec t + \tan t| \Big|_0^x = \ln|\sec x + \tan x|$$