Math 231
2/7/11

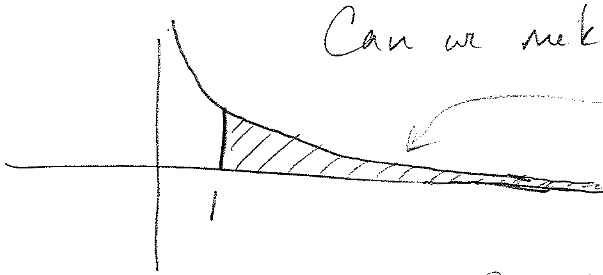
Midterm 1,
Wed Feb 16
7.1–7.5, 7.7, 7.8
8.1, 8.2.



# Improper integrals

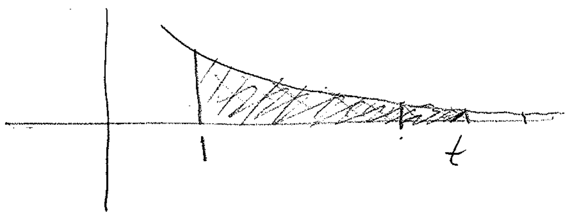
For $f(x)$ a positive function, $\int_a^b f(x)\,dx$ = area between graph and $x$ axis.

Consider $f(x) = \frac{1}{x^2} > 0$ on $[1, \infty)$



Can we make sense of area between graph and $x$-axis?

Yes. Define Area as limit of areas on larger and larger bounded intervals



$$A(t) = \int_1^t \frac{dx}{x^2}$$

$$\text{Area} = \lim_{t\to\infty} A(t) \quad \text{if exists}$$

$$= \lim_{t\to\infty} \int_1^t \frac{dx}{x^2} = \lim_{t\to\infty} -\frac{1}{x}\Big|_1^t = \lim_{t\to\infty}\left(1 - \frac{1}{t}\right) = 1.$$

In general,

$$\int_a^\infty f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx \quad \text{provided the limit exists.}$$

$$\int_{-\infty}^a f(x)\,dx = \lim_{t\to-\infty} \int_t^a f(x)\,dx \quad \text{" " " "}$$

If both exist, define $\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^a f(x)\,dx + \int_a^\infty f(x)\,dx.$

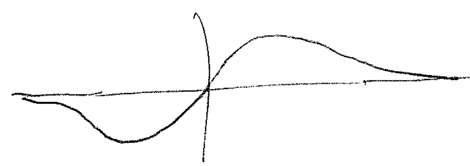
integral is <u>convergent</u> if limits exist, <u>divergent</u> otherwise. (26)

<u>Ex</u> $\displaystyle\int_1^\infty \frac{dx}{x} = \lim_{t\to\infty} \int_1^t \frac{dx}{x} = \lim_{t\to\infty} \ln|x|\Big|_1^t = \lim \ln|t| = \infty.$

so, $\displaystyle\int_1^\infty \frac{dx}{x}$ is divergent.

$\displaystyle\int_1^\infty \frac{dx}{x^2}$ is convergent.

<u>Ex</u> $\displaystyle\int_{-\infty}^\infty \frac{x\,dx}{1+x^2}$



<u>WRONG</u>: $\displaystyle\lim_{t\to\infty} \int_{-t}^t \frac{x\,dx}{1+x^2} = \lim_{t\to\infty} \frac{1}{2}\ln(1+x^2)\Big|_{-t}^t = \lim_{t\to\infty} \frac{1}{2}\left(\ln(1+t^2) - \ln(1+t^2)\right) = 0$

need to evaluate both limits individually

$$\int_{-\infty}^\infty \frac{x\,dx}{1+x^2} = \int_0^\infty \frac{x\,dx}{1+x^2} + \int_{-\infty}^0 \frac{x\,dx}{1+x^2} = \lim_{t\to\infty}\int_0^t \frac{x\,dx}{1+x^2} + \lim_{s\to-\infty}\int_s^0 \frac{x\,dx}{1+x^2}$$

$$= \lim_{t\to\infty} \frac{1}{2}\ln(1+x^2)\Big|_0^t + \lim_{s\to-\infty} \frac{1}{2}\ln(1+x^2)\Big|_s^0$$

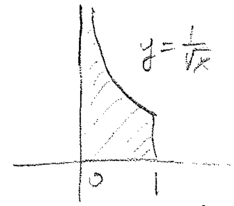
neither limit exists, so

$\displaystyle\int_{-\infty}^\infty \frac{x\,dx}{1+x^2}$ diverges.

<u>Ex</u> [student] $\displaystyle\int_{-\infty}^\infty \frac{dx}{1+x^2} = \int_0^\infty \frac{dx}{1+x^2} + \int_{-\infty}^0 \frac{dx}{1+x^2} = \lim_{t\to\infty} \tan^{-1}(x)\Big|_0^t + \lim_{s\to-\infty} \tan^{-1}(x)\Big|_s^0$

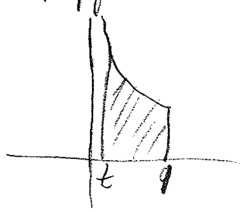
$$= \frac{\pi}{2} - 0 + 0 - \left(-\frac{\pi}{2}\right) = \pi.$$

Another improper integral:




$$\int_0^1 \frac{dx}{\sqrt{x}} = \lim_{t\to 0} \int_t^1 \frac{dx}{\sqrt{x}} = \lim_{t\to 0} 2\sqrt{x}\Big|_t^1 = \lim_{t\to 0}(2-2\sqrt{t}) = 2$$

Approximate, take a limit.




If $f$ is continuous on $[a,b)$, discontinuous (or undefined) at $b$, then

$$\int_a^b f(x)dx = \lim_{t\to b} \int_a^t f(x)dx \quad \text{provided the limit exists.}$$

In this case, $\int_a^b f(x)dx$ is <u>convergent</u>, otherwise it is <u>divergent</u>.

Similarly if $f$ continuous on $(a,b]$, discontinuous at $a$, then

$$\int_a^b f(x)dx = \lim_{t\to a} \int_t^b f(x)dx \quad \text{provided this exists ... convergent/divergent...}$$

$f$ continuous on $[a,c)\cup(c,b]$, discontinuous at $c$, then

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx \quad \text{provided both are convergent.}$$

<u>EX</u> [student]

$$\int_0^1 \frac{dx}{x^p} \quad \text{convergent for which real values of } p?$$

- $p=1 \Rightarrow \int_0^1 \frac{dx}{x} = \lim_{t\to 0} \int_t^1 \frac{dx}{x} = \lim_{t\to 0} \ln|x|\Big|_t^1 = \lim_{t\to 0} -\ln|t| = \infty$ diverge.

$p\neq 1 \Rightarrow \int_0^1 \frac{dx}{x^p} = \lim_{t\to 0} \int_t^1 \frac{dx}{x^p} = \lim_{t\to 0} \frac{x^{-p+1}}{-p+1}\Big|_t^1 = \frac{1}{-p+1} - \lim_{t\to 0} \frac{t^{-p+1}}{-p+1} = \begin{cases} \infty & \text{if } -p+1<0 \\ \frac{1}{-p+1} & \text{if } -p+1>0 \end{cases}$

so $p \geq 1$ divergent, $p < 1$ convergent

**EX** For which real values of $p$ is

$$\int_1^\infty \frac{dx}{x^p}$$ convergent?

$$p = 1 \Rightarrow \int_1^\infty \frac{dx}{x} = \lim_{t \to \infty} \ln t = \infty \text{ divergent}$$

$$p \neq 1 \Rightarrow \int_1^\infty \frac{dx}{x} = \lim_{t \to \infty}\left(\frac{1}{-p+1} - \frac{t^{-p+1}}{-p+1}\right) = \begin{cases} \infty & -p+1 > 0 \\ \frac{1}{-p+1} & -p+1 < 0 \end{cases}$$

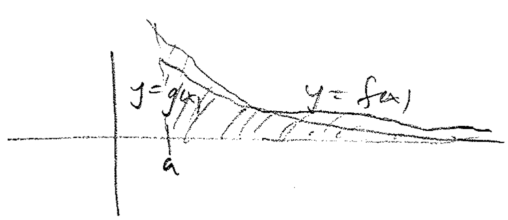
$p \leq 1$ divergent, $p > 1$ convergent

Comparison test for convergence / divergence:

If $f(x) \geq g(x) \geq 0$ for all $x \geq a$, then

- If $\int_a^\infty f(x)\,dx$ is convergent, so is $\int_a^\infty g(x)\,dx$.
- If $\int_a^\infty g(x)\,dx$ is divergent, so is $\int_a^\infty f(x)\,dx$

Picture:




bigger areas of approximates
$\Rightarrow$ bigger area.

**Ex** Even though can't compute $\int_0^\infty e^{-x^2}dx$, observe $0 < e^{-x^2} \leq e^{-x}$ for all $x$,

$$\int_0^\infty e^{-x}dx = \lim_{t \to \infty} \int_0^t e^{-x}dx = \lim_{t \to \infty} -e^{-x}\Big|_0^t = \lim_{t \to \infty} 1 - e^{-t} = 1 < \infty$$

so, $\int_0^\infty e^{-x^2}dx$ converges by comparison test.

- To use comparison test, you need to be able to compute the integral of a function bounding yours above or below

- do <u>not</u> get a value, only know convergence/divergence.

<u>EX</u> [student] decide convergence/divergence of

$$\int_1^\infty \frac{x^8+7x^5+x^2+\sqrt{2}x+1}{x^9+x^7+x^5+x^4+x}\,dx$$

observe $\dfrac{x^8+7x^5+x^2+\sqrt{2}x+1}{x^9+x^7+x^5+x^4+x} = \dfrac{1/x+\frac{7}{x^2}+\frac{1}{x^7}+\frac{\sqrt{2}}{x^8}+\frac{1}{x^9}}{1+1/x^2+1/x^4+1/x^5+1/x^8}$

$$\geq \frac{1/x}{5} = \frac{1}{5x} > 0$$

for $x \geq 1$

$\displaystyle\int_1^\infty \frac{dx}{5x} = \frac{1}{5}\underbrace{\int_1^\infty \frac{dx}{x}}_{\text{divergent}}$ by comparison test, given integral is divergent

— Moral: don't work too hard!